Edge-Unfolding Nested Polyhedral Bands

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Abstract

A *band* is the intersection of the surface of a convex polyhedron with the space between two parallel planes, as long as this space does not contain any vertices of the polyhedron. The intersection of the planes and the polyhedron produces two convex polygons. If one of these polygons contains the other in the projection orthogonal to the parallel planes, then the band is *nested*. We prove that all nested bands can be *unfolded*, by cutting along exactly one edge and folding continuously to place all faces of the band into a plane, without intersection.

Key words: polyhedra, folding, slice curves

1 Introduction

It has long been an unsolved problem to determine whether every polyhedron may be cut along edges and unfolded flat to a single, non-overlapping poly-

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gon [9,7,6]. An interesting special case emerged in the late 1990s:[1] can the *band* of surface of a convex polyhedron enclosed between parallel planes, and containing no polyhedron vertices, be unfolded without overlap by cutting an appropriate single edge? A band and its associated polyhedron are illustrated in Figure 1.

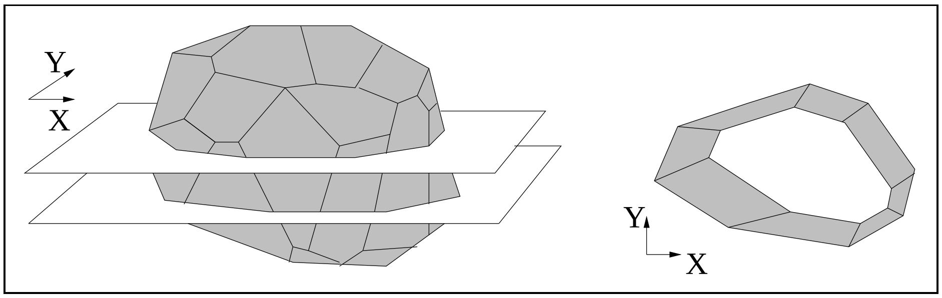

Fig. 1. A polyhedron cut by two parallel planes, and the projection of the resulting band onto the *xy* plane.

This band forms the side faces of what is known as a *prismatoid* (the convex hull of two parallel convex polygons in $\mathbb{R}^3$) but the band unfolding question ignores the top and bottom faces of the prismatoid. An example was found (by E. Demaine and A. Lubiw) that shows how flattened bands can end up overlapping if a "bad" edge is chosen to cut; see Figure 2.

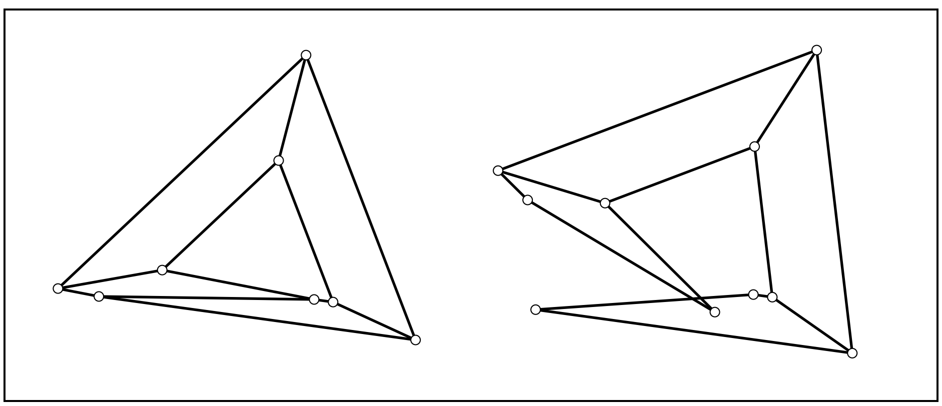

Fig. 2. A band that self-intersects when cut along the wrong edge and unfolded. Left: projection of band. Right: self-intersecting unfolding.

Band-like constructs have been studied before. Bhattacharya and Rosenfeld [3] define a polygonal *ribbon* as a finite sequence of polygons, not necessarily coplanar, such that each pair of successive polygons intersects exactly in a common side. Triangular and rectangular ribbons (both open and closed) have also been studied. Arteca and Mezey [2] deal with continuous ribbons. Simple bands can be used as linkages to transfer mechanical motion, as pointed out by Cundy and Rollett [5]. Open and closed rings of rigid panels connected by hinges have also been considered in robotics as another model for robot arms with revolute joints. For example, their singularities are well understood

[1] Posed by E. Demaine, M. Demaine, A. Lubiw, and J. O'Rourke, 1998.

mathematically [4]. As a special case of the more general *panel-and-hinge* structures studied in rigidity theory, they are relevant to protein modeling [11]. In all these instances, almost no attention was paid to questions regarding their non-self-intersecting states or their self-collision-avoiding motions.

There is one unfolding result that is particularly relevant to our problem, which may be interpreted as unfolding infinitely thin bands. This result states that a "slice curve," the intersection of a plane with a convex polyhedron, develops (unfolds) in the plane without overlap [8]. This result holds regardless of where the curve is cut. Thus, both the top and the bottom boundary of any band (and in fact any slice curve between), will not self-intersect after a band has been flattened. So overlap can only occur from interaction with the cut edge, as in Figure 2.

Here we will prove that a particular type of band can be unfolded by explicitly identifying an edge to be cut. A band is *nested* if projecting the top polygonal rim A orthogonally onto the plane of the bottom polygonal rim B results in a polygon nested inside B. For example, the band in Figure 1 is nested. Intuitively, we might expect to obtain a nested band if both parallel planes cut the polyhedron near its "top". We prove that all nested bands can be unfolded. Our proof provides more than non-overlap in the final planar state: it ensures non-intersection throughout a continuous unfolding motion.

2 Bands

We first define bands more formally and analyze their combinatorial and geometric structure, without regard to unfolding.

Consider a convex polyhedron P, and let $z_0, z_1, \dots, z_m$ denote the sorted z coordinates of the vertices of P. Pick two z coordinates z_A and z_B that fall strictly between two consecutive vertices z_i and z_{i+1},and suppose z_A is above z_B: $z_i < z_B < z_A < z_{i+1}$. The *band* determined by P, z_A, and z_B is the intersection of P's surface with the horizontal slab of points whose z coordinates satisfy $z_B \le z \le z_A$.

The band is a polyhedral surface with two components of boundary, called A and B. Specifically, A is the *top polygonal rim* of the band, i.e., the intersection of P's surface with the plane $z = z_A$, and B is the *bottom polygonal rim*, corresponding to the plane $z = z_B$. Both rims A and B are convex polygons in their respective planes, being slice curves of a convex polyhedral surface P. All vertices of the band are vertices of either A or B.

Every vertex of the band is incident to exactly three edges. two along the rim

A or B containing the vertex, and the third connecting to the other rim. This third edge, called a *hinge*, is part of an edge of the original polyhedron P connecting a vertex of P with z coordinate less than z_B to a vertex of P with z coordinate greater than z_A. The hinge from each vertex of the band defines a perfect matching between vertices of the top rim A and vertices of the bottom rim B. This matching is consistent with the cyclic orders of A and B in the sense that, if vertex a_i of A is paired with vertex b_i of B, then the vertex a_{i+1} clockwise around A from a_i is paired with the vertex b_{i+1} clockwise around B from b_i. This correspondence defines a consistent clockwise labeling of the vertices $a_0, a_1, \ldots, a_{n-1}$ of A and the vertices $b_0, b_1, \ldots, b_{n-1}$ of B, unique up to a common cyclic shift [2] .

Each face of the band is a quadrilateral spanned by two adjacent vertices a_i and a_{i+1} on the top rim A and their corresponding vertices b_i and b_{i+1} on the bottom rim B. This facial structure follows from the edge structure of the band. Each face corresponds to a portion of a face of the original polyhedron P (so in particular it is planar). Because edges $a_i a_{i+1}$ and $b_i b_{i+1}$ lie in a common plane as well as in parallel horizontal planes, the edges themselves must be parallel. Thus every face of the band is in fact a trapezoid, with parallel top and bottom edges.

3 Nested Bands

Next we analyze the geometric structure of nested bands in particular, still without regard to unfolding.

A band is *nested* if the orthogonal projection of A into the xy plane is contained inside the orthogonal projection of B into the xy plane. (Of course, a band is just as nested if instead B's projection is contained inside A's projection, but in that case we just reflect the band through the xy plane.)

Nested bands have a particularly simple structure when projected into the xy plane. As with all bands, each face projects to a trapezoid. The unique property of a nested band is that none of its edges cross in projection. This property follows because the projected edges are a subset of a triangulation of the projections of A and B, which themselves do not intersect by the nested property. (In non-nested bands, edges of A intersect edges of B in projection.) Thus the projected trapezoidal faces of the band form a planar decomposition of the region of the xy plane interior to the projection of B and exterior to the projection of A.

[2] Throughout this paper, indices are taken modulo n.

In the xy projection, the *normal cone* of a vertex a_i of the rim A (or more generally any convex polygon) is the region between the two exterior rays that start at a_i and are perpendicular to the incident edges $a_{i-1}a_i$ and a_ia_{i+1} respectively. See Figure 3.

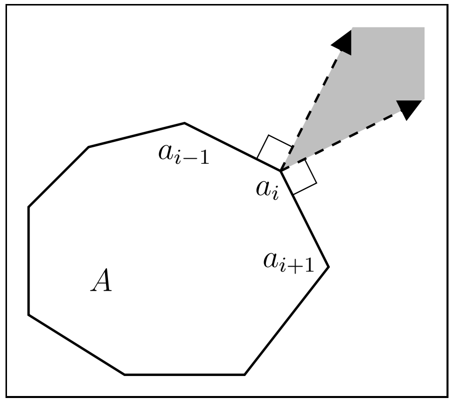

Fig. 3. The normal cone of a vertex a_i.

Lemma 1 *In the projection of a nested band, not all hinges a_ib_i can be to the left (or all to the right) of the normal cones of their inner endpoint a_i.*

PROOF. The following proof refers exclusively to the xy projection. Suppose without loss of generality that all hinges are clockwise of their respective normal cones on the inner polygon A. For each i, define T_i to be the trapezoid with vertices $a_{i-1}, a_i, b_{i-1}, b_i$, and let h_i denote its height, i.e., the distance between the opposite parallel edges $a_{i-1}a_i$ and $b_{i-1}b_i$. See Figure 4. Because a_ib_i is right of the perpendicular at a_i to a_ia_{i+1}, and because the interior angle at b_i is convex, the convex angle $a_ib_ib_{i-1}$ is less than the convex angle $a_{i+1}a_ib_i$. Thus, the height h_i of T_i is less than the height h_{i+1} of the clockwise next trapezoid T_{i+1}. Applying this argument to every T_i, we obtain a cycle of strict inequalities $h_0 < h_1 < \cdots < h_{n-1} < h_0$, which is a contradiction. $\square$

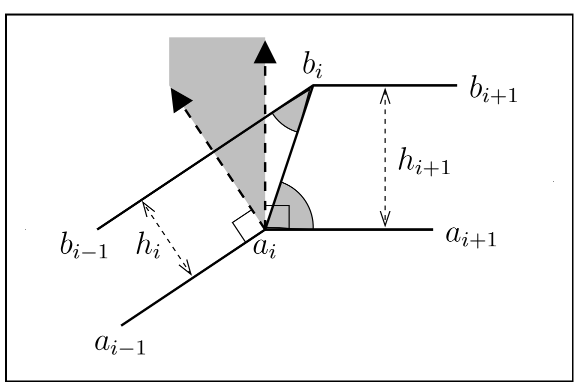

Fig. 4. If the hinge a_ib_i is right of the normal cone at a_i, then the top shaded angle is less than the bottom shaded angle, so $h_i < h_{i+1}$.

4 Unfolding Nested Bands

After cutting a single hinge, a *flattening motion* is a continuous motion during which each face moves rigidly but remains connected to each adjacent face via their common hinge, and the final configuration is planar. If no intersection occurs during the motion, then this motion is a *continuous unfolding*. If the resulting configuration is non-self-intersecting, but intersection occurs during the motion, then we call the motion an *instantaneous unfolding* and the resulting configuration an *unfolded state*. Thus in Figure 2 we would say that the band has been flattened, but because it self-intersects it has not unfolded.

We now describe the particular flattening motion that will lead to our unfolding, though it requires some effort to prove non-intersection particularly of the final state. The flattening motion is based on *squeezing* together the two parallel planes $z = z_A$ and $z = z_B$ that contain the rims A and B, keeping the planes parallel and keeping each rim chain on its respective plane. At time t, the squeezing motion reduces the vertical separation between the two parallel planes down to $(1-t)(z_A - z_B)$, that is, it linearly interpolates the separation from the original $z_A - z_B$ down to 0.

The squeezing uniquely determines the hinge dihedral angles necessary to keep the vertices of the band on their respective moving planes (assuming exactly one edge of the band has been cut). See Figure 5 for an example of the projected motion. For nested bands, the motion increases the interior angle at every vertex of each chain in projection. This property can be seen by examining any two adjacent faces that are being "squeezed". Both faces rotate continuously to become more horizontal. If we force one of the faces to keep its vertices in the parallel planes, but allow the second face to only follow this motion rigidly (i.e., the dihedral angle at the hinge remains fixed), then we see that the edges of the second face would no longer be on the horizontal planes. To compensate, the second face must perform a (dihedral) rotation about the hinge. In fact the interior angle at the hinge must increase, causing the interior angles of the rims to increase.

Furthermore, because the interior angle at a vertex of a nested band can open only to π, the opening chains cannot self-intersect after such a motion (a fact already known from the slice-curve result mentioned earlier). For the same reason, an opening chain will always have only right turns.

As the parallel planes squeeze together, each band face remains a trapezoid in the projection. Edges $a_i a_{i+1}$ and $b_i b_{i+1}$ remain parallel and retain their original lengths throughout. Hinge projections lengthen as the band is squeezed, which causes the trapezoid angles to change. Because b_i and b_{i+1} move orthogonally away from $a_i a_{i+1}$, acute trapezoid angles increase toward $\pi/2$ and

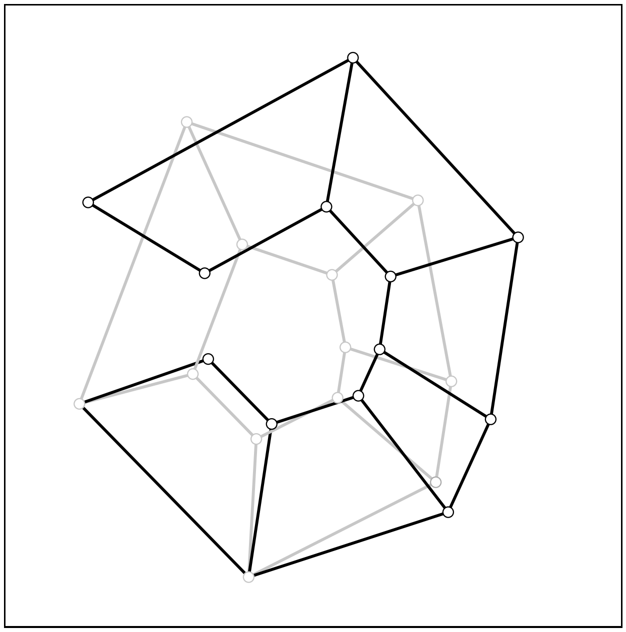

Fig. 5. A view from above of a nested band during a squeezing motion.

obtuse angles decrease toward $\pi/2$.

The goal of this section is to show that the band does not self-intersect if we cut a specific hinge. We mention that self-intersection of the band in 3D implies self-intersection in projection, so it suffices to prove that there is no self-intersection in projection to establish that there is no self-intersection in 3D. It turns out that the only cause of self-intersection is the cut edge, to which we now turn our attention.

Suppose that we cut hinge $a_i b_i$ and hold $a_{i-1}a_i$ fixed along the x-axis in the positive direction. The motion separates two copies of a_i; we call the stationary one a_i, and call the moving one a^*. Correspondingly, for the outer polygon, the direction of $b_{i-1}b_i$ remains fixed (it moves away from $a_{i-1}a_i$ because the trapezoid enlarges in projection, but remains parallel) and b^* is a "moving" endpoint. Thus the cut hinge is split into edges $a_i b_i$ and $a^* b^*$. See Figure 6.

We now introduce some basic terminology and properties of an opening chain in the projection. Given a chain with all right turns, the *interior angle* α_i at a vertex a_i is the angle $a_{i-1}a_i a_{i+1}$ located on the right side of a_i. Let $\tau_i = \pi - \alpha_i$ be the *turn angle* at a_i. Let θ_j be the counterclockwise angle of the vector $a_j - a_{j-1}$ from the positive x-axis. If $a_i - a_{i-1}$ is fixed along the positive x-axis, then for a chain with all right turns we have: $\theta_i = 0$, $\theta_{i-1} = \tau_{i-1}$, and in general,

$$\theta_{i-k} = \sum_{j=i-k}^{i-1} \tau_j. \qquad (1)$$

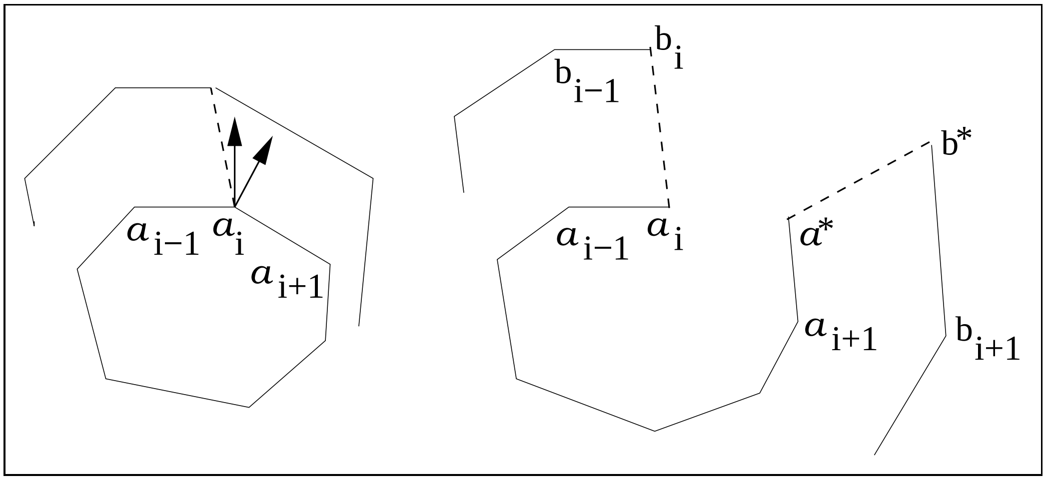

Fig. 6. Left: projection of the inner convex chain and part of the outer chain. Hinge $a_i b_i$ and the cone of vertex a_i are shown. Right: the result of cutting at $a_i b_i$ and flattening.

We define three classes of shapes that a chain with only right turns may have: convex, weakly convex, and spiral. Refer to Figure 7. A chain is *convex* if joining the endpoints with a closing segment yields a convex polygon. A chain is *weakly convex* if joining the endpoints with a closing segment yields a simple polygon with no exterior angles smaller than $\pi/2$. If a chain is not convex or weakly convex, it is a *spiral*.

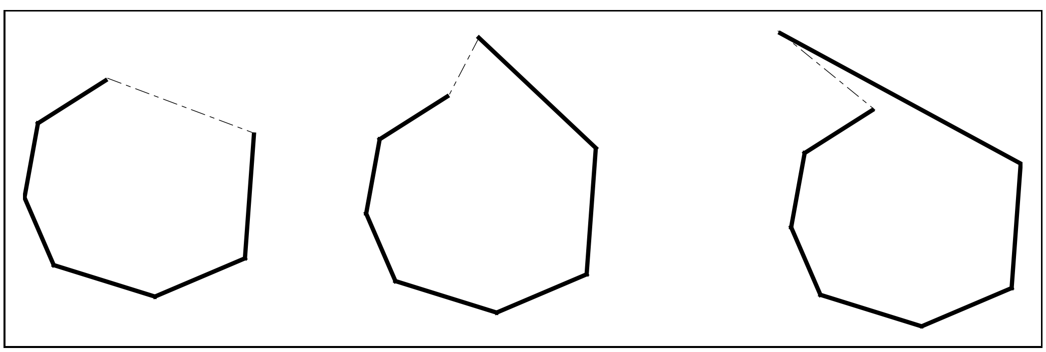

Fig. 7. Types of chains, from left to right: convex, weakly convex, spiral. Endpoints are joined by dashed line segments.

Our results below apply to the projection of a band after it has been partially squeezed, and also to a band that has been flattened.

Lemma 2 *Flattening a band cannot produce an inner chain that is a spiral.*

PROOF. Consider the line that passes through a_i and is orthogonal to $a_i a_{i+1}$ prior to any motion. Let R be the halfplane defined by this line, and which does not contain the normal cone at a_i. Let Q be the halfplane to the right of $a_{i-1}a_i$. See Figure 8(a).

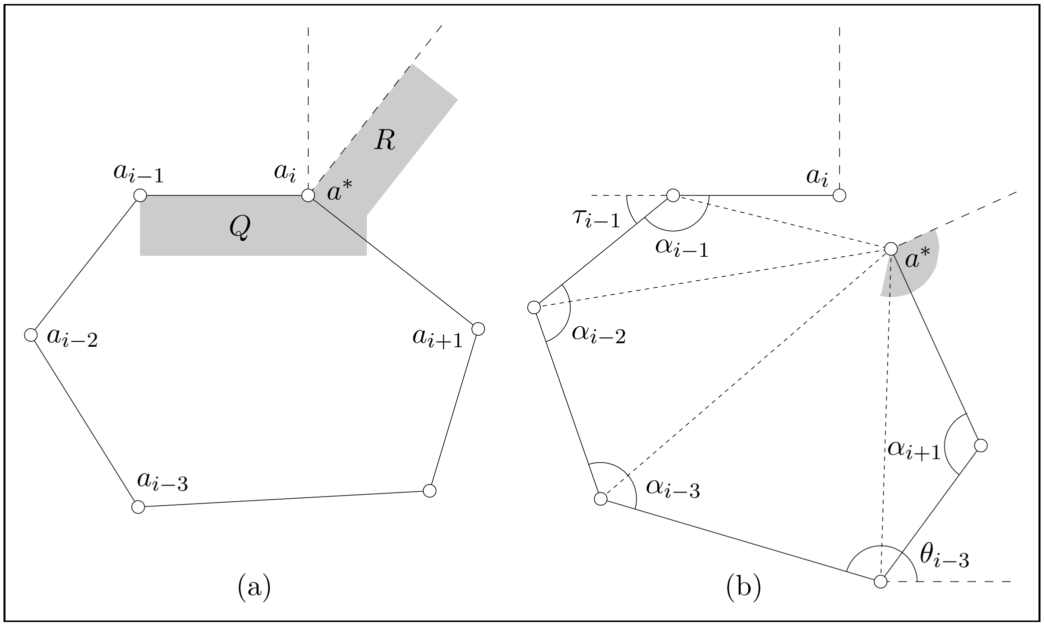

Fig. 8. (a) Cut at $a_i = a^*$ ensures that a^* moves into $R \cup Q$; (b) After partial opening motion. The shaded cone at a^* shows the range of instantaneous vector displacements caused by rotations at all vertices.

We establish two claims about the effect of any opening motion:

(1) Every edge $a_k a_{k+1}$ of A turns clockwise in the sense that the vector $a_{k+1} - a_k$ turns clockwise. In particular, $a^* a_{i+1}$ turns clockwise around a^*.
(2) The vertices of A always lie in $R \cup Q$.

By the second claim, a^* cannot reach a position where it will be on the hull of a spiral, as shown on the right of Figure 7. By symmetry, this is enough to prove that no spiral can be created.

The motion of a^* is determined by the opening of the internal angles of all vertices on the counterclockwise path from a_{i-1} to a_{i+1}. Specifically, the opening of an angle α_j at a vertex a_j causes a^* to rotate instantaneously about this vertex a_j. Because all internal angles α_j increase, all respective turn angles τ_j decrease. Equation 1 shows that, if all turn angles τ_j decrease, then θ_{i+1} decreases, which corresponds to a clockwise turn of $a^* a_{i+1}$. This establishes the first claim, for the same holds true for all θ_k and thus all edges $a_k a_{k+1}$.

Turning to the second claim, we consider that initially each rotation centered at a vertex a_k creates a vector displacement of a^* perpendicular to $a_k a^*$, which aims inside $R \cup Q$. More importantly, no vector points through the cone at a_i. Once a^* moves, the clockwise turn of $a^* a_{i+1}$ guarantees that again no vector displacement points through the cone. More specifically, a displacement can either point in to a proper subset of $R \cup Q$ as shown in Figure 8(b), or, it can point through the extension of $a_i a_{i-1}$, but only if a_k is above and to the right of a^*. In any case, a^* can never enter the cone at a_i. Thus, a^* remains inside $R \cup Q$ throughout any opening motion. The same holds true for all vertices of A. □

By Lemma 2, we can now assume that the inner chain of a flattened nested band is either convex or weakly convex.

Lemma 3 *If a flattened band overlaps, then some edge crosses over one of the two cut hinge copies.*

PROOF. The result mentioned earlier on slice curves shows that neither the A nor the B chain can self-intersect. Chain A cannot intersect B if they are in convex position: Let ℓ be the line that passes through segment $a_i a^*$. This is a supporting line of chain A. Now we can sweep a line parallel to ℓ within the halfplane containing A. At any position of the sweep line we obtain a segment that is in the interior of the band, then a segment between points on A, and finally another band segment. Thus no part of B can intersect a part of A.

Now suppose that A is weakly convex, and without loss of generality let a^* be on the hull. Thus a^* is to the left of the line ℓ_2 that extends through $a_{i-1}a_i$. A cannot cross B in the halfplane to the right of ℓ_2, by the same arguments given above. Consider the halfplane to the left of ℓ_2. We know that $b_{i-1}b_i$ is parallel to ℓ_2. The remaining edges of B are to the left of $a_i b_i$, and within the strip formed by ℓ_2 and the supporting line of $b_{i-1}b_i$. Thus there is no way for a part of A to intersect these edges without also intersecting $a_i b_i$. Therefore band overlap requires one or the other chain to cross through $a_i b_i$ or through $a^* b^*$ (cf. Figure 2). □

Lemma 4 *Let $T_1 = a_{i-1}a_i b_i b_{i-1}$ and $T_2 = a_i a_{i+1} b_{i+1} b_i$ be two adjacent trapezoids of a band. If a cut is made at their common hinge $a_i b_i$, then hinge copy $a^* b^*$ of T_2 must rotate clockwise with respect to hinge copy $a_i b_i$ of T_1.*

PROOF. Hinge $a_i b_i$ rotates only because T_1 is flattened. Hinge $a^* b^*$ rotates because T_2 is flattened, but also because all interior angles of chain A open. We first examine the two rotations caused by the flattening of T_1 and T_2. If we were to flatten these two trapezoids without actually cutting their common hinge, then the two hinge copies, $a^* b^*$ and $a_i b_i$ (still glued together), would both undergo an identical rotation. This would also cause $a^* a_{i+1}$ to rotate counterclockwise. Instead, by imposing that $a^* a_{i+1}$ maintains its orientation, and that a^* remains incident to a_i, we see that $a^* b^*$ must rotate clockwise with respect to $a_i b_i$. Now we may apply the third source of rotation to $a^* b^*$, which must also be clockwise because $a^* a_{i+1}$ rotates clockwise, by the first claim of Lemma 2. □

Lemma 5 *If a flattening motion produces an inner chain A that is convex, then the flattened band does not overlap, i.e., it has an unfolded state.*

PROOF. Suppose that we cut at a_ib_i. If the inner chain is convex, then all of chain A has moved to the right of $a_{i-1}a_i$ which means that chain A cannot intersect a_ib_i. By Lemma 4, a^*b^* has undergone an additional clockwise rotation with respect to a_ib_i. This relative difference in orientation and the convexity of the opened chain imply that the two hinges cannot intersect, and that b^* (and in fact, all sections of the B chain) cannot cross over a_ib_i. By symmetric arguments, nothing crosses over a^*b^*. □

Lemma 6 *If we cut a hinge a_ib_i that is inside the normal cone of a_i, then the nested band can be continuously unfolded.*

PROOF. Apply the squeezing motion to flatten the nested band. If the opened inner chain is convex at any moment, then there is no intersection by Lemma 5. Now suppose that at some instant the chain is weakly convex, and without loss of generality a^* is on the hull. Let ℓ be a fixed line through a_i and the original position of b_i. Let the direction of ℓ be from a_i to b_i. Because a_ib_i can only rotate so that it becomes more orthogonal to $a_{i-1}a_i$, it will always remain within the normal cone, and more specifically, to the left of ℓ. From claim (2) of Lemma 2, a^* never enters the interior of the normal cone, but rather moves immediately into $R \cup Q$, where it remains throughout the motion. Hinge copy a^*b^* must rotate clockwise, because it becomes more orthogonal to a^*a_{i+1} and because all interior angles open. Thus given the relative positions of a_i and a^*, the two hinges cannot intersect.

From the arguments above, hinge copy a^*b^* cannot cross over ℓ in the region above $a_{i-1}a_i$. Since a_ib_i was in the normal cone of a_i, angle $a_{i+1}a^*b^*$ is obtuse. Thus a^*b^* cannot rotate enough clockwise to intersect ℓ in the region below $a_{i-1}a_i$.

The intersection of ℓ with the band produces one line segment, $a_\ell b_\ell$ (other than the point a_i). The line ℓ divides the band into two disjoint bands whose interior chain is convex, and so by the arguments of Lemma 3, they do not self-intersect.

By symmetric arguments there can be no intersection if the partially opened chain is weakly convex with a_i on the hull. □

Note that Lemma 6 implies that any band with an acute interior angle can be unfolded, for there must be a hinge inside the normal cone at an acute angle: see Figure 9. Similarly, a geodesic cut (not necessarily along a hinge) perpendicular to supporting lines to both the inner and outer chains leads to a continuous unfolding.

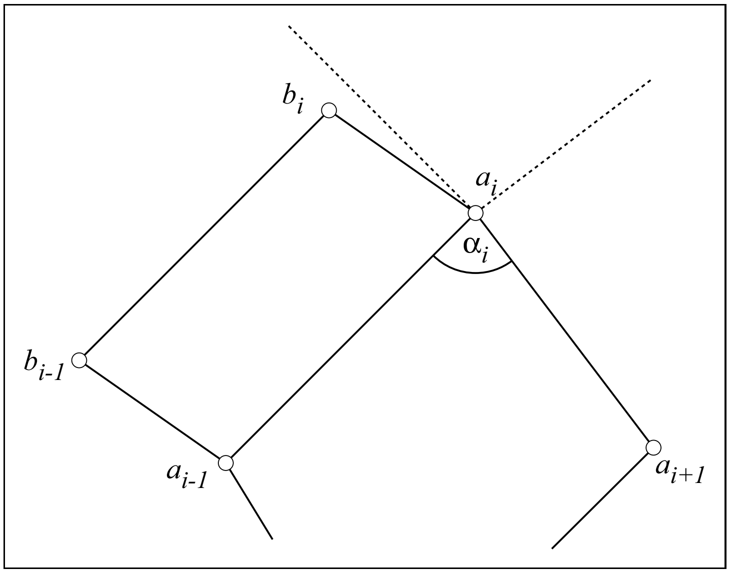

Fig. 9. $\alpha_i < \pi/2$. If the hinge a_ib_i is outside the normal cone, it is impossible to complete a trapezoid whose base is parallel to a_ia_{i+1}.

We now characterize the types of chains that may be obtained after cutting at a_ib_i and flattening (or partially flattening by applying a squeezing motion). We say that a chain is "safe" if it could not be the inner chain of a self-intersecting band. Thus, by Lemma 5, all convex chains are safe. Weakly convex chains are not necessarily safe. Let us subdivide this class of chains into *L-weakly convex* and *R-weakly convex*, depending on which endpoint is on the hull (clearly exactly one of the two endpoints must be on the hull). If a_i is on the hull then the chain is L-weakly convex. Otherwise, if a^* is on the hull, the chain is R-weakly convex. A chain may open to a weakly convex position and be safe, as seen in Lemma 6. In fact, for an R-weakly convex chain, an intersection cannot occur if a_ib_i was initially to the left of the normal cone at a_i. In this case, a^* must be to the right of a_ib_i. Since a^*b^* rotates more clockwise than a_ib_i, no intersection can occur (we may form a line through the final position of a_ib_i and repeat the arguments of Lemma 6. So we say that a chain is "unsafe" if it is R-weakly convex and in the initial projection a_ib_i was to the right of the normal cone at a_i (see Figure 10). By symmetry a chain is unsafe if it is L-weakly convex and in the initial projection a_ib_i was to the left of the normal cone. We note that even under these conditions there may be no intersection at any time during a flattening motion. In other words, the term "unsafe" serves just as a warning.

Clearly if a band has no unfolded state then all vertices are associated with unsafe openings. By Lemma 1, somewhere there is a vertex a_k whose hinge is counterclockwise of the normal cone at a_k, while the hinge at a_{k+1} is clockwise

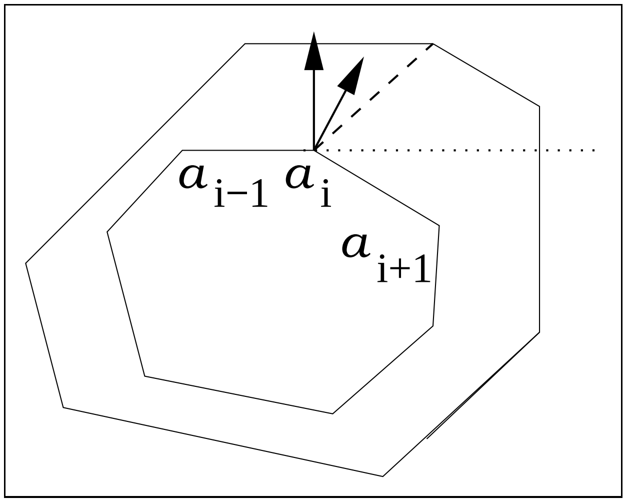

Fig. 10. After cutting at a_i, the inner chain will become R-weakly convex if a^* ends up above the dotted line. In this case the cut is labeled "unsafe" if hinge a_ib_i (shown dashed) is to the right of its normal cone.

of its respective cone. For the cuts at both vertices to produce unsafe inner chains, cutting at a_k must produce an L-weakly convex chain, while cutting at a_{k+1} must produce an R-weakly convex chain (see Figure 11).

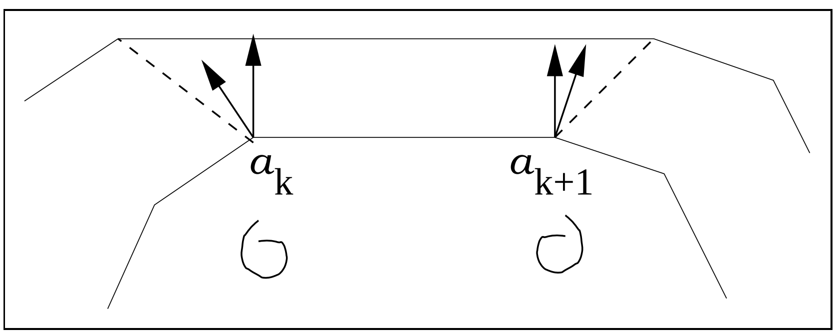

Fig. 11. Two successive vertices, a_k and a_{k+1}, whose cuts produce different weakly convex chains (indicated by the curves below the vertices).

Lemma 7 *At least one of the cuts at a_k or a_{k+1} (defined above) must result in an opened chain that is safe.*

PROOF. Let us begin by cutting at a_{k+1} and flattening. Hold a_ka_{k+1} fixed and open all angles. Assume that the opened chain is unsafe. This means that newly created a^* must end up in the upper-right quadrant of a_{k+1}. Now we make a new cut at a_k, and translate the entire opened chain (except the fixed edge) so that a^* re-attaches to a_{k+1}. We let the translated copy of a_k retain its label, and call the fixed edge a^*a_{k+1}. Notice that a_k must be in the lower left quadrant of a^* (see Figure 12).

Now we have a new opened chain, except that we have not taken care of the openings at the angles of a_k and a_{k+1}. Because $a_{k+1}a_{k+2}$ (previously a^*a_{k+2})

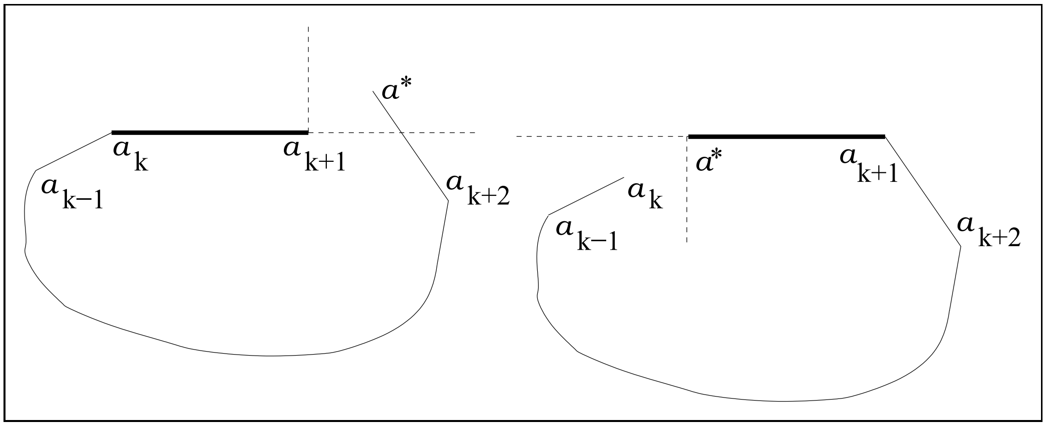

Fig. 12. Left: an opened chain. Right: translating part of the chain so that the cut vertex is switched. This is a new opened chain, except for the angle at a_{k+1}.

had rotated clockwise in the first opening, and we have merely translated it back, we must rotate it counterclockwise to return it to its initial orientation. We must then further rotate it counterclockwise in order to open the interior angle at a_{k+1}. The entire chain will rotate rigidly as well. Thus a_k cannot cross into the upper-left quadrant of a^*. Now notice that during the first opening, edge $a_{k-1}a_k$ rotated clockwise, due to the opening of the angle at a_k. So we might expect that in order to compensate for this in our final diagram we should rotate $a_{k-1}a_k$ counterclockwise (which might cause a_k to go above the horizontal line). After all, if a cut is made at a_k, then $a_{k-1}a_k$ must rotate counterclockwise from its initial position, but now it is clockwise. However, because the opening of the angle at a_{k-1} was included in the first opening, and this has not been tampered with, then edge $a_{k-1}a_k$ must be in its correct position. The counterclockwise motion produced by adjusting the angle at a_{k+1} is enough to make the direction of $a_{k-1}a_k$ more counterclockwise than it was initially. We conclude that cutting at a_k leads to either an R-weakly convex opening or to a convex opening. Therefore cutting at a_k cannot be unsafe. □

Because we can always find a vertex to cut so that the inner chain opens to a position that is safe, we can always find an edge to cut along so that a nested band has an unfolded state.

Theorem 8 *Every nested band has an unfolded state.*

Theorem 9 *For every nested band there exists a continuous unfolding motion.*

PROOF. Consider the squeezing motion that we have defined. Parameterize any point p on the band by its original height z_p divided by the height z of the original band. After partially squeezing the band to height z_S , the new height of p will be $z_S \frac{z_p}{z}$. So all the points with the same original height have the same new height at some time of the squeezing motion.

Now consider two points p and q, that intersect at some instant during a squeezing motion. The two points are in some horizontal plane H when they intersect. Because p and q both started out in another horizontal plane H^*, and had the same height at all times throughout squeezing, they were part of a developing slice curve. In other words, initially there was a slice curve in plane H^*, containing p and q. All points in this curve remain coplanar (specifically in a horizontal plane) during squeezing. By the slice-curve result mentioned in the introduction, p and q can never intersect. We conclude that no intersection can occur until the final flattened configuration of the band, which is a singularity where the above arguments do not apply. However, by Theorem 8 we know that some cut exists that produces an unfolded state. Therefore by making the same cut and applying the squeezing motion, we obtain a continuous unfolding of the band. □

5 Remarks

We note that another natural continuous unfolding motion exists, that consists of $n-1$ "peeling" moves. After cutting a hinge which produces an unfolded state, we begin by performing a dihedral rotation about its neighboring hinge, so that two trapezoids become coplanar. Subsequent moves are simple dihedral rotations about successive hinges, and every time one more trapezoid is added into the coplanar subset. We omit the proof of correctness.

We believe that our results extend to non-nested bands, and also to those bands which contain polyhedron vertices on the two parallel planes. These extensions will appear in [1].

Even with it established that arbitrary bands may be unfolded without overlap, it remains interesting to see if this can lead to an unfolding of prismatoids without overlap, including the top and bottom faces. It is natural to hope that they could be nestled on opposite sides of the unfolded band, but it is not obvious how to ensure non-overlap.

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