# Single-Vertex Origami and Spherical Expansive Motions

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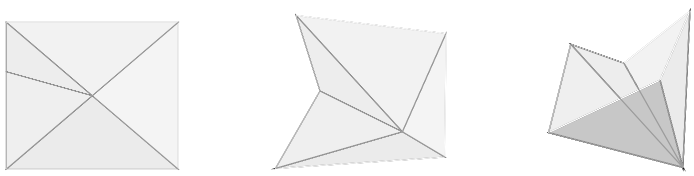
**Abstract.** We prove that all single-vertex origami shapes are reachable from the open flat state via simple, non-crossing motions. We also consider *conical* paper, where the total sum of the cone angles centered at the origami vertex is not $2\pi$. For an angle sum less than $2\pi$, the configuration space of origami shapes compatible with the given metric has two components, and within each component, a shape can always be reconfigured via simple (non-crossing) motions. Such a reconfiguration may not always be possible for an angle sum larger than $2\pi$.
The proofs rely on natural extensions to the sphere of planar Euclidean rigidity results regarding the existence and combinatorial characterization of expansive motions. In particular, we extend the concept of a pseudo-triangulation from the Euclidean to the spherical case. As a consequence, we formulate a set of *necessary conditions* that must be satisfied by three-dimensional generalizations of pointed pseudo-triangulations.

## 1 Introduction

Imagine making creases in a flat sheet of paper, all of them originating at a single vertex, and then folding along the creases *without tearing, stretching or bending the paper*, to obtain a three dimensional origami shape. Assume that the planar regions bounded by creases behave more like metal sheets than paper, i.e. they move *rigidly*, and do not go through each other during the motion. See Fig. 1. The *single-vertex origami problem* asks: *are there origami shapes which are compatible with the creases and the induced metric of the paper, but which cannot be* folded *by such a process?*



(a) (b) (c)

**Fig. 1.** A single-vertex origami fold: (a) the creased sheet of paper; (b, c) two of its possible folded shapes.

* Supported by NSF grants CCR-0105507 and NSF-DARPA CARGO-0310661.
** Supported by grants from NSERC (Canada) and NIH (US).

Here, we answer this question and generalize it in several ways. We prove that all single-vertex origami shapes can be folded from the flat state. This implies that the configuration space of all origami shapes with the same crease pattern is *connected*: any shape can be reconfigured into any other shape, via simple (non-self-intersecting) motions. We begin by formulating the problem in terms of conical *panel-and-hinge structures*, which have incident hinge-axes and in terms of *spherical polygonal linkages*. If the spherical perimeter (defined below) is no more than $2\pi$, we show that then they can be reconfigured to a spherically convex configuration. But if the perimeter is more than $2\pi$, then the configuration space may be disconnected. We leave open the question of whether simple spherical polygonal *paths* of length between $\pi$ and $2\pi$, with *short* links (less than $\pi$) is connected or not.

Our proofs rely on a generalization to the sphere of the planar *Carpenter's Rule Problem*, which asks whether every simple planar polygon can be unfolded to a convex position, in such a way that the edges maintain their lengths and do not cross throughout the motion. An *expansive motion* never decreases any interdistance between two points, therefore no collisions may occur. The two solutions of the Carpenter's Rule problem for simple planar polygonal linkages [4, 17] rely on expansive motions. In dimension two, the infinitesimal expansive motions are well understood. They form a polyhedral cone [15] whose extreme rays have a combinatorial characterization, given by pointed pseudo-triangulation mechanisms [17]. Their three-dimensional counterparts also form a polyhedral cone, defined by similar linear inequalities, but finding a combinatorial interpretation for its rays has so far remained elusive.

In this paper, we also initiate the study of expansive motions on the sphere and in 3d and give the first provable classes of spherical and 3d expansive mechanisms with one-degree-of-freedom: hemispherical pseudo-triangulations, resp. pointed *cone pseudo-triangulations*. By applying them to the spherical Carpenter's Rule Problem of perimeter $\leq 2\pi$, we obtain the proof that all *simple* folds can be opened to a spherical convex position.

*Historical background.* Computational origami is a relatively recent endeavor, see [6] for a survey. The mathematical and computational origami literature addresses questions of feasibility, characterization and NP-hardness of flat-folds, as well as applications, see e.g. [13, 1, 7]. According to Tom Hull [10], only two published articles deal with the mathematics of rigid origami [9, 14].

The topology of the configuration space for spherical linkages and single-vertex origami folds (allowing self-crossings) is studied in [12, 11].

Polygonal linkages in the plane have received a lot of attention in recent years. Relevant for our paper are the previously mentioned results on the Carpenter's Rule Problem using expansive motions [4] and pseudo-triangulations [17]. To the best of our knowledge, there are no other results studying or applying spherical or 3d expansive motions.

Rigidity for non-Euclidean geometries (including spherical and hyperbolic) is the topic of an unpublished paper of the second author [16]. The projective connections between motions of cones, motions on spheres, and motions in plane projections go back to [19]. We achieve the extension of the Carpenter's Rule problem to spherical polygons and single-vertex origami folds via spherical geometry techniques, building on ideas from [19, 16] but handling signed motions (expansive and contractive).

## 2 Definitions and Preliminaries

For rigidity theoretic terminology and concepts, we refer the reader to the classical monograph [8] and the handbook chapter [20]. For pseudo-triangulations, see [17] and [15].

*Frameworks in two and three dimensions.* A bar-and-joint framework (simply, a framework) $G(p)$ is a graph $G = (V, E)$, $V = [n] := \{1, \cdots, n\}$, embedded on a set of points $p = \{p_1, \cdots, p_n\}$, $p_i \in R^d$. In this paper $d = 2$ or $d = 3$. A pair of indices $(i, j) \in [n]^2$ may be denoted simply as $ij$. The embedded edges (segments) $p_i p_j$ are also called *bars*, and their endpoints $p_i$ are called *joints*. When the underlying graph is a path or a cycle, the framework is called a *chain* (open, resp. closed).

*Infinitesimal rigidity of frameworks.* An *infinitesimal motion* of a framework $G(p)$ is a set of velocity vectors $v = \{v_1, \cdots, v_n\}$, $v_i \in R^d$ preserving the lengths of the bars:

$$\langle p_i - p_j, v_i - v_j \rangle = 0, \forall ij \in E$$

An infimitesimal motion is *trivial* if it is a rigid transformation of the whole space. A framework is *infinitesimally rigid* if it has only trivial infinitesimal motions, and *infinitesimally flexible* otherwise.

A *flex* or motion of a framework is a set of continuous point trajectories

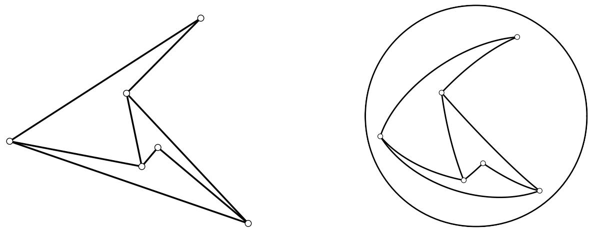
$$p(t) = \{p_1(t), \cdots, p_n(t)\}$$

which preserve the edge lengths $l_{ij}^2 = \langle p_i - p_j, p_i - p_j \rangle, \forall ij \in E$ of the initial framework $G(p) = G(p(0))$ at any moment in time $t$:

$$\langle p_i(t) - p_j(t), p_i(t) - p_j(t) \rangle = l_{ij}^2, \forall ij \in E$$

*Expansion and contraction.* Given a point set $p$ and infinitesimal velocities $v$, let us denote by $\varepsilon_{ij}$ the quantity $\varepsilon_{ij}(p, v) := \langle p_i - p_j, v_i - v_j \rangle$. For a pair $ij$ of indices (not necessarily an edge of a graph $G$), we say that the diagonal $ij$ *expands* if $\varepsilon_{ij} > 0$, *contracts* if $\varepsilon_{ij} < 0$ or is *frozen* if $\varepsilon_{ij} = 0$. An infinitesimal motion $v$ of $G(p)$ is *expansive* (resp. *contractive*) if all the non-frozen diagonals expand (resp. contract). A framework is infinitesimally *expansive* if it is infinitesimally flexible and supports a non-trivial infinitesimally expansive motion.

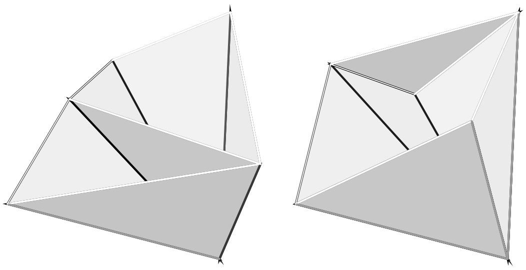
*Pointed pseudo-triangulations and mechanisms.* A special class of planar frameworks are those with no crossing edges and where every vertex is *pointed*: incident to an angle larger than $\pi$. Such frameworks are planar graph embeddings and can have at most $2n - 3$ edges. When they have the maximum number of edges, the outer face is convex and all the internal faces are *pseudo-triangles*: simple polygons with exactly three inner convex angles. As frameworks, pointed pseudo-triangulations are *minimally rigid*. Removing any edge makes then *flexible* mechanisms, with one degree of freedom. If the removed edge is a convex hull edge, the mechanism is *expansive*: the unique motion that increases the length of the removed convex hull edge, never decreases any distance between a pair of points. See [17] and Fig. 2(a).



(a) (b)

**Fig. 2.** A pointed pseudo-triangulation mechanism (a) in the plane, and (b) on the sphere.

*Panel and hinge structures.* A bounded *panel* is a simple polygon embedded flat in 3d, and intended to behave like a rigid object, i.e. to remain flat and to maintain its metric properties (edge lengths and angles). The simplest case is a triangle. We will also work with unbounded panels which are wedges defined by two semi-infinite rays.



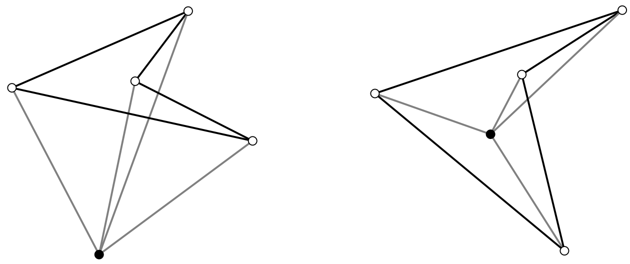
(a) (b)

**Fig. 3.** Examples of panel-and-hinge structures. (a) Some hinges may be incident with several panels. (b) A triangulated polyhedral surface is a special case of a panel and hinge structure.

A *hinge* is a line segment or ray common to at least two panels. In this paper we consider only hinges that appear as complete bounding edges of their incident panels. In particular, a hinge does not run through the interior of a panel and does not extend beyond its boundary. The panels are attached rigidly to hinges: they are allowed to rotate about hinges, but not to slide along them.

A *panel-and-hinge* structure is a collection of panels connected via hinges. Examples include those in Fig. 1, where all hinges are concurrent, and Fig. 3, where each hinge is incident to two panels.

*Conical panel-and-hinge structures.* In this paper we work only with *conical panel-and-hinge structures*, in which all the hinges are concurrent to a unique vertex called the *cone vertex*, as in Fig. 3(a). In this case, each panel contains at most two hinges. The conical structures are *bounded*, if all panels are bounded polygons and the hinges are line segments, as in Fig. 3(a), or *unbounded*, when the hinges are infinite rays and the panels are wedges. For the purpose of this paper, in the bounded case it suffices to work only with triangular faces. We distinguish two cases, *pointed* and *non-pointed conical structures*, depending on whether the cone vertex is pointed (all incident segments are contained in a half-space defined by a plane through the cone vertex) or not. See Fig. 4.



(a) (b)

**Fig. 4.** (a) A pointed and (b) a non-pointed conical framework arising from conical panel-and-hinge structures. The cone vertex is black.

*From conical panel-and-hinge structures to bar-and-joint frameworks.* To each bounded conical panel-and-hinge structure we associate a bar-and-joint framework $G_0(p)$ in 3d. The vertices of $G_0$ correspond to those of the conical structure and the edges to sides of the triangular panels. The cone-vertex is always labeled with 0, and is incident with all the other vertices, labeled from 1 to $n$. $G_0$ is embedded as $G_0(p)$ on the set of points bounding the triangular panels, $p_i \in R^3$, $i = 0, 1, \cdots, n$. Such a bar-and-joint framework is called a *cone framework.*

An *infinitesimal motion* of a bounded conical panel-and-hinge structure is an infinitesimal motion of the associated conical bar-and-joint framework.

For unbounded conical panel-and-hinge structures, we first bound the panels by *cutting* them into triangles. Then we define the infinitesimal motion of the bounded structure. The following straightforward lemma shows that the motion can be uniquely extended to all the points of the original unbounded panels.

**Lemma 1.** *(***Infinitesimal velocities of arbitrary points)** *Any point $p'$ situated on the supporting line of a hinge $p_0p_i$ or on the supporting plane of a panel $p_0p_ip_j$ can be assigned an infinitesimal velocity $v'$ uniquely determined by the velocities of the points bounding the hinge or panel.*

*Spherical frameworks.* If we intersect an unbounded conical structure with a sphere centered at the cone vertex, we obtain a *spherical framework.* It consists of *spherical bars* along great-circle arcs on the sphere, and joints which are the intersection of the hinges with the sphere. The incidence structure of a spherical framework is a graph $G$ obtained from $G_0$ by deleting the cone-vertex 0 and its incident edges. The remaining graph $G$ has the vertex set $[n]$. The *length* of a spherical edge $ij$ is the size of the angle $\angle p_ip_0p_j$ centered at the cone vertex.

A spherical framework fully contained in a hemisphere is called *hemispherical.* It arises from a pointed conical panel-and-hinge structure. See Fig. 5.

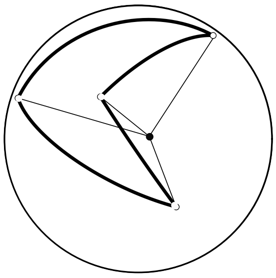


**Fig. 5.** A hemispherical framework, and the corresponding pointed conical panel-and-hinge structure. The cone-vertex (black) is the center of the sphere.

## 3 Infinitesimal rigidity of conical and spherical frameworks

In this section we show how to associate to every *planar framework* a 3d pointed *conical framework*, and therefore also a hemispherical framework and a pointed conical panel-and-hinge structure, and vice-versa. The association preserves infinitesimal rigidity and maintains the expansion pattern.

*The cone of a planar framework.* Let $G(p)$ be a planar framework embedded in $R^2$, viewed as the affine plane $z = 1$ in $R^3$. The *cone* over $G$ is the graph $G_0 = (V_0, E_0)$, $V_0 = V \cup \{0\}$ (0 is the *cone vertex*) and $E_0 = E \cup \{0i\}_{i \in V}$. The *canonical conical framework* $G_0(p)$ over $G(p)$ with center $p_0 \in R^3$ is an embedding of the cone $G_0$ over $G$ so that it extends $G(p)$ and embeds the cone vertex at the center $p_0$. In this paper we take $p_0 = 0$. More generally, a cone framework $G_0(q)$ over $G(p)$ will have the cone vertex at $q_0$ and the $i$th vertex $q_i$ on the line $p_0 p_i$.

*Infinitesimal motions of planar and conical frameworks.* The association we just described induces a *projection map* carrying a conical framework[3] $G_0(q)$ with no points on the plane $z = 0$ into a framework $G(p)$ in $R^2$ embedded in the plane $z = 1$ (and vice-versa). We now turn our attention to establishing and proving the connection between infinitesimal motions of a planar framework and the associated conical framework.

Let $G_0(q)$ be a conical framework with the cone vertex at the origin, $q = (q_0, q_1, \cdots, q_n)$, $q_0 = 0$ and $q_i = (x_i, y_i, z_i), z_i \neq 0$. Let $u = (u_0, u_1, \cdots, u_n)$, $u_0 = 0$ be an infinitesimal motion of the framework $G_0(q)$ in $R^3$, with the cone vertex pinned down. Let $\mathbf{e}$ the unit vector along the $z$-axis. See Fig. 6.

**Proposition 1.** *Consider the map $R^3 \mapsto R^2$ taking the points $q_i = (x_i, y_i, z_i)$ on the sphere to points $p_i = (\frac{x_i}{z_i}, \frac{y_i}{z_i}, 1)$ in the $z = 1$ plane and velocities $u_i$ to:*

$$v_i = \frac{1}{z_i} \left( u_i - \langle u_i, \boldsymbol{e} \rangle \boldsymbol{e} \right)$$

*Then $u = (0, u_1, \cdots, u_n)$ is an infinitesimal motion of the conical framework $G_0(q)$ in $R^3$ iff $v = (v_1, \cdots, v_n)$ is an infinitesimal motion in $R^2$ of the framework $G(p)$.*

[3] In this section, we denote by $p$ a planar point set and by $q$ a 3d point set.

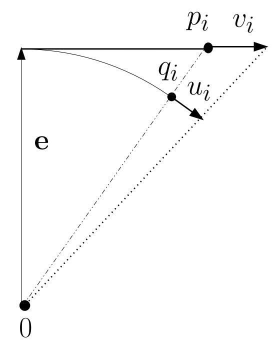



**Fig. 6.** The projection map, shown here in a section along the plane spanned by a point $q_i$ on the sphere and the $z$-axis. It takes a point $q_i$ on the sphere to a point $p_i$ in the $z = 1$ plane, and an infinitesimal velocity $u_i$ of $q_i$ to an infinitesimal velocity $v_i$ of $p_i$.

*If two points $q_i$ and $q_j$ are in the same hemisphere ($z > 0$ or $z < 0$), then $u$ is expansive for the pair $q_i, q_j$ iff $v$ is expansive on $p_i, p_j$ in the plane. If $q_i$ and $q_j$ are on opposite hemispheres, then $u$ is expansive for the pair $q_i, q_j$ iff $v$ is contractive on $p_i, p_j$ in the plane.*

**Proof:** The bars from the cone vertex give the equations $\langle q_i - q_0, u_i - u_0\rangle = \langle q_i, u_i\rangle = 0$. Applying them on $i$ and $j$, the expansion values become:

$$\langle p_i - p_j, v_i - v_j\rangle = \langle \frac{q_i}{z_i} - \frac{q_j}{z_j}, \frac{1}{z_i}(u_i - \langle u_i, \mathbf{e}\rangle \mathbf{e}) - \frac{1}{z_j}(u_j - \langle u_j, \mathbf{e}\rangle \mathbf{e}) =$$

$$= -\frac{\langle q_i, u_j\rangle + \langle q_j, u_i\rangle}{z_i z_j} + [\frac{\langle u_i, \mathbf{e}\rangle}{z_i} - \frac{\langle u_j, \mathbf{e}\rangle}{z_j} - \frac{\langle u_i, \mathbf{e}\rangle}{z_i} + \frac{\langle u_j, \mathbf{e}\rangle}{z_j}] =$$

$$= -\frac{\langle q_i, u_j\rangle + \langle q_j, u_i\rangle}{\langle \mathbf{e}, q_i\rangle\langle \mathbf{e}, q_j\rangle} + \frac{1}{z_i z_j}[\langle q_i, u_i\rangle - \langle q_i, u_j\rangle - \langle q_j, u_i\rangle\langle q_j, u_j\rangle] = \frac{1}{z_i z_j}\langle q_i - q_j, u_i - u_j\rangle$$

Thus the sign (positive, negative or zero) of $\langle p_i - p_j, v_i - v_j\rangle$ is the same as the sign of $\langle q_i - q_j, u_i - u_j\rangle$ when the two points $q_i$ and $q_j$ are in the same hemisphere ($z_i z_j > 0$) and opposite when in different hemispheres ($z_i z_j < 0$). □

As a simple corollary we obtain:

**Lemma 2.** *Let $G(p)$ be a planar framework and let $G_0(p)$ be an associated 3-dimensional cone framework. Then:*

1. *If $G(p)$ is infinitesimally rigid, then so is $G_0(p)$.*
2. *If $G(p)$ is infinitesimally expansive, and the cone vertex is pointed, then $G_0(p)$ is also expansive.*

If the planar framework $G(p)$ is a pseudo-triangulation mechanism, then the spherical framework associated to the cone framework $G_0(p)$ will be called a *hemispherical pointed pseudo-triangulation mechanism.* See Fig. 2(b). Interpreting Lemma 2 in this case leads to:

**Lemma 3.** *A hemispherical pseudo-triangulation mechanism is expansive.*

As another corollary we also obtain the following simple necessary condition for 3d expansive motions. Given a graph $G$ and a vertex $i$, let $V_i$ be the vertex $i$ together with the set of neighbors of $i$. The $i$th *star* $G_i$ of $G$ is the subgraph induced on $V_i$.

**Lemma 4.** *Let $G(q)$ be a spatial flexible framework and let $G_i(q)$ be the conical framework induced on the star $G_i$. If a conical framework $G_i(q)$ is not infinitesimally expansive, then $G(q)$ is not infinitesimally expansive. In other words, the following two local pointedness conditions are necessary for infinitesimal expansiveness:*

1. *The cone vertex of each star $G_i(q)$ is pointed (in 3d), and*
2. *Each induced cone framework projects to a* pointed *and* non-crossing *2d framework.*

Part 2 of the previous lemma relies on the existence of expansive motions for planar pointed graphs (which are subgraphs of pointed pseudo-triangulations), see [17, 15]. Lemma 4 gives a necessary condition for a 3d framework to be expansive, and thus precludes any rigidity-theoretic generalization of planar pointed pseudo-triangulation mechanisms to 3d.

## 4 Convexifying spherical polygonal linkages

A spherical framework is a graph $G$ embedded on the surface of a sphere, with edges going along arcs of great circles. The *length* of an edge $q_iq_j$ is the angle between the two line segments joining the center of the sphere to the two endpoints of the edge. To simplify the presentation, we will assume that edges have lengths different from 0 and $\pi$, i.e the endpoints of every edge are distinct and not-antipodal (the results hold even without this assumption).

A great-circle cuts the sphere into two hemispheres. With respect to a hemisphere, the defining great-circle is called its *equator*, and the point where the normal to the equator plane crosses the hemisphere is called the *pole* of the hemisphere.

A hemispherical framework is one lying on a hemisphere. A *proper* framework has all its edge lengths at most $\pi$. Of special interest are spherical and hemispherical polygons and polygonal paths. The perimeter of a polygon and the length of a polygonal path is the total sum of its edge lengths. A hemispherical polygon is *convex* if the projection on the plane going through the pole and parallel to the plane of the equator is a convex polygon.

**Theorem 1.** *Every simple hemispherical polygon of perimeter $\leq 2\pi$ and every simple hemispherical polygonal path of length $\leq \pi$ can be unfolded to a convex polygon using expansive motions.*

**Proof:** Note that any polygon of perimeter at most $2\pi$ must lie inside a closed hemisphere. If it lies inside an open hemisphere, we proceed with the following case. We return below to the extreme case, when there are antipodal points.

Project the polygon in an open hemisphere to a plane tangent to the pole of the hemisphere. Find an expansive infinitesimal motion of the planar polygon, e.g. one induced by a pointed pseudo-triangulation mechanism as in [17], or an arbitrary one obtained by a linear program as in [4]. Lemmas 2 and 3 imply that the induced cone framework is also moving expansively.

Move the spherical mechanism until an alignment event occurs, as in [17]. Notice that the alignment of two incident edges in a spherical pseudo-triangulation mechanism happens exactly when the projection on the $z = 1$ plane has the corresponding edges aligned (but beware, the projection does not move as a planar rigid framework). The combinatorial structure of the spherical and planar pseudo-triangulations is the same. Therefore, as long as the polygon stays inside a hemisphere, there is always an expanding motion given by a pseudo-triangulation mechanism, and the convexification algorithm of [17] applies identically.

There is a concern that the motion might leave an open hemisphere. This occurs when two antipodal points appear. With two antipodal points in a polygon, there must be two paths joining the points, each of length at least $\pi$. This implies that the polygon must have length exactly $2\pi$, and the two paths must each be great circle segments joining the antipodal points. In this situation, we may have the entire polygon on a single great circle, and we are finished with a flat spherical polygon. The alternative is that we have a 'wedge' formed by two intersecting great circles. Freezing one, we can rotate the other to make them flat, on opposite segments of a great circle. This motion is obviously expansive. Finally, we note that if we did not start with antipodal points, we can only achieve this configuration at an alignment event, so our previous process is complete and only requires a check for a wedge after each alignment event.

Finally, we note that if the algorithm terminates with a convex polygon which is not flat, then we have a *convex cone* from the center of the sphere which has a total angle of less than $2\pi$. This means the spherical polygon had perimeter less than $2\pi$. Therefore, any polygon of perimeter $2\pi$ will terminate with a flat configuration on a single great circle.

The polygonal path of length not more than $2\pi$ can be reduced to the closed polygon case by joining the endpoints of the path via a geodesic spherical path (which stays inside the hemisphere and bends at vertices and edges of the polygonal path). The total length of the added geodesic edges is at most the length of the polygon, so we have reduced to the previous case. Moreover, for any path less than $\pi$ we could choose the additional edges to make the total length exactly $2\pi$ and terminate with a flat (collinear) polygonal path. If the length is exactly $\pi$, a limiting argument guarantees we can also achieve flatness. □

Let's emphasize as separate corollaries two ideas from the previous proof.

**Lemma 5.** *A spherical polygon of perimeter $2\pi$ can be unfolded to a great circle using expansive motions.*

**Corollary 1.** *If the geodesic line added between the endpoints of a spherical polygonal path doesn't increase the total length to more than $2\pi$, then the polygonal path can be made flat on a great circle using expansive motions.*

The case not covered by the previous corollary, the hemispherical polygonal path (even when its total length is under $2\pi$) may need contractive (or partially contractive) motions to convexify. We conjecture that the convexification is always possible.

*Conjecture 1.* Every spherical polygonal path of length at most $2\pi$ can be flattened without collisions onto a great circle.

Just as in the planar case, we obtain:

**Corollary 2.** *Two similarly oriented configurations of a simple spherical polygon with fixed edge lengths and perimeter $\leq 2\pi$ lie in the same component of the configuration space.*

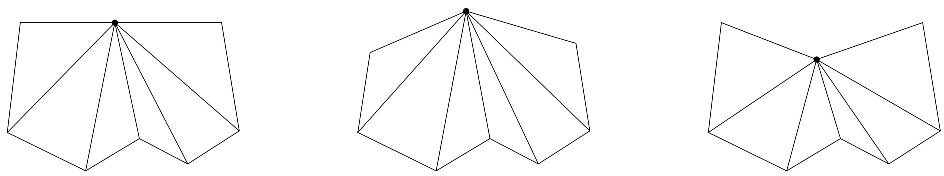
Indeed, one can move each one into convex position and then reverse one unfolding path to achieve the reconfiguring motion.

What happens for spherical polygons and paths longer than $2\pi$? They obviously cannot be convexified, since they do not fit on a great circle. The question is whether they can always be reconfigured to any other position. The following simple example shows that this may not be always possible.

*Example 1.* Consider a spherical quadrilateral with all edge lengths equal to $\frac{2\pi}{3} - \epsilon_i$, for four distinct values $\epsilon_i$. Its perimeter is $\frac{8\pi}{3} - \sum \epsilon_i > 2\pi$, and any consecutive pair of edges adds to over $\pi$ in length. The framework fits into a hemisphere in two ways, which cannot be reconfigured one to another without self-intersections.

Moreover, even when a reconfiguration os such spherical linkages is possible, a combination of expansive and contractive motions may have to be used.

## 5 Unfolding single-vertex origami



(a) (b) (c)

**Fig. 7.** Single-vertex origami with the fold-vertex on the boundary of the piece of paper: (a) on an edge, (b) at a convex corner and (c) at a reflex corner.

A single-vertex origami is a creased piece of paper with all the creases incident to one vertex. Assume first that the vertex lies in the interior of the piece of paper. For simplicity, assume that the paper has a polygonal boundary with one vertex on each crease line. The total angular length of the corresponding spherical polygon or path framework is $2\pi$ in this case. We also consider the case when the polygonal piece of paper has the fold-vertex situated on a boundary edge or at a corner of the paper. The total angular length of the corresponding spherical polygon or path framework is $\pi$ in the first case, and it can be either strictly less than or larger than $2\pi$ in the second case, depending on the corner angle being convex or reflex. See Fig. 7. Theorem 1 implies:

**Corollary 3.** *Every simple single-vertex origami fold with the fold-vertex interior to the paper or interior to a boundary edge or situated at a convex vertex can be unfolded with expansive motions.*

**Corollary 4.** *The configuration space of simple single-vertex origamis with the fold-vertex interior to the paper, or to a boundary edge or situated at a convex vertex is connected. Two shapes can be reconfigured one into the other with simple, non-self-intersecting motions.*

Conjecture 1 extends to single-vertex origamis with the fold-vertex situated on a reflex corner of the paper.

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