Remarks on the Voronoi Game

Jihui Zhao
Computer Science
Rutgers University

William Steiger
Computer Science
Rutgers University

September 23, 2006

1 Introduction and Summary

In the Voronoi game two players A and B each play n points in a region $S \subset R^d$ of bounded Lebesgue measure μ, and obtain scores equal to the measure of the union of their Voronoi regions. In the *one round* version, A places all his points and then B places hers. In the *alternating* version A plays and then B plays, in each of the n successive rounds.

For $d = 1$, first studied by Ahn et. al. [1], an illustrative case is $S = S^1$, the unit circle in R^2 (or any simple closed curve). In the one-round version B can acheive $\mu(B) = \mu(A)$ by the simple strategy of placing each of her points between a different pair of adjacent points of A, thereby winning half of each interval. In fact unless A plays correctly (by evenly spacing his n points), B can win by (i) placing two points in the largest of A's intervals (arbitrarily close to the endpoints), (ii) placing one point arbitrarily in each of the remaining intervals (except the smallest), and (iii) avoiding the smallest of A's intervals; nearly all the largest interval is clearly greater than half the average of the largest and the smallest, and she gets half of each of the other intervals.

If S is "cut" and opened into a finite interval, or segment, these same observations can be used to show that now A can always win the one-round version by playing the "critical" (evenly spaced) points; $1/(2n), 3/(2n), \ldots, (2n-1)/(2n)$ along the length of S. Also, B can always win the alternating game (by getting as many critical points as possible and then playing cleverly).

For $d = 2$ Cheong et. al. [2] took S to be a square of area one, and studied the one-round version, where the situation differs completely from the $d = 1$ segment game. Now A need not win because there are no "critical" points to guarantee victory. In fact, using a beautiful probabilistic argument they proved that for $n > n_0$, B has a winning strategy. An interesting followup paper by Fekete and Meijer [5] explored the continuum between a square and a segment by allowing S to be a rectangle of base x and height $1/x$. As $x \geq 1$ increases, S becomes more "linear", more like a segment. At a certain point, the behaviour of the one round Voronoi game on $S \subset R^2$ switches to be that of the game on a segment: once $x \geq \sqrt{n/2}$, $n \geq 3$, there are again critical points which, if A takes them in his move, will guarantee victory.

Nothing seems to be known about the alternating game in dimension $d > 1$, though it seems likely that B should win if $n > 1$. By examining the few simple cases we prove that

Lemma 1 *For $n = 2$ and $n = 3$, B has a winning strategy on any bounded convex set $S \subset R^2$.*

The $n = 3$ instance shows that the winning player need not lead until the end, making it difficult to imagine using induction.

Returning to the one-round game where $S = \{(x,y) : x,y \in [0,1]\}$ is the unit square let us also imagine it as a torus to avoid boundary effects. If $n = k^2$ and A plays his n points at $S_A = \{(\frac{2i-1}{2k}, \frac{2j-1}{2k}),\ i,j = 1,\ldots,k\}$. The Voronoi region of a point of S_A is a square of area $1/n$ within which, any point played by B would only control an area of $1/(2n) - \varepsilon$. Nevertheless we can prove

Lemma 2 *If B plays her n points at $S_B = \{(\frac{12i-6+(-1)^{(j+1)}}{12k}, \frac{2j-1}{2k}) : i,j \in 1,\ldots,k\}$, she gets .518 area, and this value is a local maximum.*

In view of the fact that B always has a winning strategy, it is interesting to think about $A's$ best move. Cheong et. al [2] suspected its the equilateral triangular lattice (S. Har-Peled, pers. com.). In fact they established the remarkable fact that given *any* n points in S the expected area of the Voronoi region of a point placed uniformly at random is greater than $1/(2n)$ (the average of A's n regions is $1/n$). Along these lines we can prove

Theorem 1 *Suppose B places a point uniformly at random in S after A has played n points. The expected area of her Voronoi region is minimized when the first player places his points on the equilateral triangular lattice.*

Here we are assuming n is large so Voronoi regions of the lattice points nearest the boundary contribute less than $c/\sqrt{n}$.

When A plays all n points on the triangular lattice in S, each points Voronoi region (not touching the boundary) is a regular hexagon H_i of area h. It is easy to see that

Lemma 3 *If B plays a point P on the boundary of some H_i or on an edge of the lattice, $\mu(Vor(P)) = h/2$.*

Also

Lemma 4 *The maximal area of the Voronoi region for a single point played by B is $(0.51273\ldots)h$ (it's a local maximum).*

References

[1] H. Ahn, S. Chen, O. Cheong, M. Golin, and R. vanOostrum. Competitive Facility Location: The Voronoi Game. *Theoretical Computer Science 310*,457-467 (2004).

[2] O. Cheong, S. Har-Peled, N. Linial, and J. Matoušek. The One-Round Voronoi Game. *Discrete & Computational Geometry 31*, 125-138, 2004.

[3] O. Cheong, A. Efrat, and S. Har-Peled. On Finding a Guard that Sees Most and a Shop that Sells Most. *Proc.*15^{th} *SIAM Symp. on Discr. Algs*, (2004)

[4] F. Dehne,R. Klein,and R. Seidel. Maximizing a Voronoi Region: The Convex Case. *Proc.* 13^{th} *SIAM Symp. on Discr. Algs* 624-634, (2002)

[5] S. Fekete and H. Meijer. The One-Round Voronoio Game Replayed. *Lecture Notes in Computer Science, Vol. 2748*, 150-161, Springer Verlag (2003).