Delaunay terminal edge algorithm

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The Lepp Delaunay terminal edge method was introduced in a rather intuitive basis as a generalization of previous longest edge algorithms in [1, 3]. This was supported by the improvement properties of both the longest edge bisection of triangles [4] and the Delaunay algorithm, and by using a bound on the largest angle for Delaunay terminal triangles. Later in [2] we discussed some geometrical properties including some (rare) potential loop cases for angle tolerance greater than 22°. However, while empirical studies show that the method behaves analogously to the circumcircle method in 2-dimensions [2, 3], formal proofs on algorithm termination and on optimal size property have not been established due to the difficulty of the analysis. In this work we improve and extend the geometrical results and prove algorithm termination.

1 Concepts and algorithm

Terminal edge and Lepp concepts. An edge E is called a terminal edge if this is the longest edge of every triangle that shares E (one or two triangles depending if E is a boundary or an interior edge in the mesh respectively). In addition for any triangle t, the longest edge propagation path, Lepp(t), is defined as a finite sequence of edge-neighbor triangles such that each triangle is a neighbor by the longest edge of its proceding triangle in the sequence. Thus the last longest edge in the sequence is a terminal edge in the mesh.
Algorithm The quality triangulation algorithm can be simply described as follows: Starting with a constrained Delaunay triangulation of an input PSLG geometry, the set S of bad quality triangles is found (triangles for which its smallest angle is less than an input angle tolerance α_{tol}). Then for each triangle t in S, and repeatedly, while t remains in the mesh, the following steps are performed: (1) Lepp(t) and associated terminal triangles t_1, t_2 and terminal edge l are found; (2) the midpoint M of l is Delaunay inserted in the mesh unless t_1 or t_2 has a constrained second longest edge l, whose midpoint is Delaunay inserted in the mesh otherwise.
Assumption In order to simplify the analysis we assume that $\alpha_{tol} \leq \alpha_{limit} \approx 22.2°$ to avoid a rare unfrequent looping case related with some constrained items [2].
Longest edge bisection. For the longest edge bisection of any arbitrary triangle t, defined as the splitting of t by the midpoint of its longest edge into triangles t_B, t_A (see Figure 1), the following angle bounds hold: $\alpha_1 \geq \alpha_0/2$, $\beta_2 \geq 3\alpha_0/2$; while that if t is obtuse, then $\alpha_1 \geq \alpha_0$ and $\beta_2 \geq 2\alpha_0$.
Delaunay terminal triangles. For a Delaunay mesh, an unconstrained terminal edge has the following property [1, 3]: For any pair of Delaunay terminal-triangles t_1, t_2 sharing a non-constrained terminal edge, largest angle$(t_i) \leq 2\pi/3$ for $i = 1, 2$.

2 New results on the algorithm

Since the longest edge bisection of the terminal triangles can be seen as a first step in the Delaunay insertion of a terminal edge midpoint M, in this paper we study: (1) The geometrical properties of

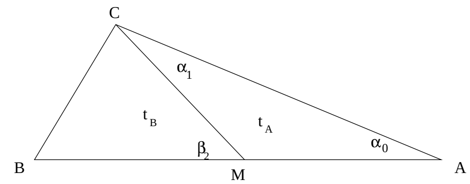

Figure 1:

the longest edge bisection of a Delaunay terminal triangle which allows to obtain improved angle bounds for the bisected triangles t_A, t_B in Figure 1, as well as bounds on the distance from M to previous vertices; (2) some improvement properties associated to the elimination of obtuse bad triangle t_A when this remains in the mesh after the Delaunay insertion of M.
To this end we firstly study a characterization of Delaunay terminal triangles. Consider a Delaunay terminal triangle with a fixed second longest edge CA, longest edge BA and smallest angle at vertex A (see Figure 2). Then the vertex B and the terminal edge midpoint M respectively belong to regions EFC and $E'F'N$, where the straight line constraint CE identifies triangles with largest angle equal to $2\pi/3$, and the arc constraints EF and EG follow from the fact that BA is a terminal edge.

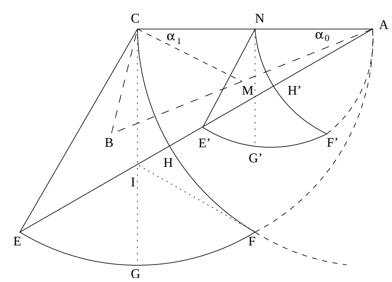

Figure 2: Regions EFC and E'F'N are geometrical places for vertex B and midpoint M for a terminal triangle BAC with respective smallest and largest angles of vertices A and C

We use the characterization of the diagram of Figure 2 to obtain improved bounds on the angles α_1 and α_2 for acute bad quality terminal triangles. We also obtain bounds on the distance of the terminal edge midpoint M to the previous vertices in the mesh.

Finally, if after the Delaunay insertion of a terminal edge midpoint M, a bad obtuse triangle MAC (see Figure 1) remains in the mesh, we prove that after a finite number of point insertions, inside or in the neighborbood of the circumcircle $CC(MAC)$, a significant discrete angle improvement is achieved. We also prove that only a finite sequence of worsening or slightly improving bad obtuse triangles rarely appear in the mesh. By using the preceding results and assuming that the $PSLG$ geometry does not include any constrained angle less than $\pi/2$ (analogously to the Ruppert condition), we prove algorithm termination.

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