Computing Convex Hulls on Beckenbach and Drandell Geometries

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Abstract

We take Beckenbach and Drandell geometries as concrete models within the framework of complete ordered incidence geometries and show that it is possible to extend Kirkpatrick and Seidel's convex hull algorithm to these two models within essentially the same $O(n \log h)$ time complexity.

1 A historical note and summary

Peixoto's seminal work [5] in generalizing the notion of convex sets in the plane to the Beckenbach geometry, where lines are graphs of functions (along with vertical lines) – under certain fairly weak conditions – was later outgrown by Drandell [3] who considered more general families of curves. Both of these papers seem essentially unreferenced in the literature and apparently unknown even to Ben-Tal and Ben-Israel who, in [2], develop a very nice axiomatic approach to establish a geometric foundation for convexity theory based on affine sets. The *ordered incidence geometry* (OIG) and the more interesting *complete*-OIG (COIG) are the ultimate outcome of this axiomatization.

Considering the extensive literature on algorithms for convexity problems, it is somewhat unexpected to be unable to find results ample enough to circumscribe the full generality of the COIG. This surprise increases in face of the mathematical beauty of this geometry and can only be appeased by the apparent scarcity of real world applications. On the other hand, this should not deter mathematicians and computer scientists from advancing on seemingly theoretical areas.

It is the intent of this paper to illustrate our work on extending computational geometry algorithms, in particular convexity algorithms, from the Euclidean plane to models of COIG, namely, the two dimensional Beckenbach geometry and the more general Drandell geometry.

To that effect, we show that Kirkpatrick and Seidel's $O(n \log h)$ algorithm [4] for constructing the convex hull of a planar set S of n points can be thus extended, while keeping the same time complexity, up to a factor that depends only on the complexity of the lines of the geometry. Here, h denotes the number of vertices of $\mathrm{Hull}(S)$.

2 Beckenbach and Drandell Geometries

Most of this background material has been adapted to render a uniform notation from [1], [2], [3] and [6].

Definition 1 ([1]) *Let (a, b), (c, d) be open intervals (not necessarily bounded) in $\mathbb{R}$, and let $X = (a, b) \times (c, d)$. A family $\mathcal{F}$ of continuous functions $F : (a, b) \to (c, d)$ is a* Beckenbach family *if for any two points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2) \in X$ with $x_1 < x_2$, there exists a unique function in $\mathcal{F}$, denoted $F_{p_1 p_2}$, whose graph passes through p_1 and p_2.*

Definition 2 ([2]) *Let (a, b), (c, d), X and $\mathcal{F}$ be as above. The* Beckenbach geometry *$\mathcal{G}_{\mathcal{F}}$, determined by $\mathcal{F}$, is a two dimensional geometry over the set X where the 1-affine through any pair of points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ in X is*

1. *the graph of $F_{p_1 p_2}$ (defined above) if $x_1 \neq x_2$, or*
2. *the vertical line $x = x_1$ if $x_1 = x_2$ (also denoted $F_{p_1 p_2}$).*

Examples of Beckenbach geometries are:

- the planar Euclidean geometry — or its restriction to any open rectangle;
- the $\mathcal{G}_{\mathcal{F}}$ geometry obtained when we define $\mathcal{F}$ to be all the translations of a fixed parabola.

More generally, we could take a Beckenbach family $\mathcal{F}$ to be comprised of functions of the form $f(\alpha, x) - \beta$, where $f(\alpha, x)$ is differentiable in α for all x and $\partial f / \partial \alpha$ is a strictly monotone function of x. Hence, the two previous examples are $\mathcal{F} = \{\alpha x - \beta\}$, and $\mathcal{F} = \{k(x + \alpha)^2 - \beta\}$, for fixed $k \neq 0$, respectively.

Besides Beckenbach geometries, COIGs include the Drandell geometry.

Definition 3 ([3]) *The* Drandell geometry *is comprised of the points in $\mathbb{R}^2$ together with a family of curves $\mathcal{F}$ in $\mathbb{R}^2 \cup \{\omega\}$ satisfying*[1]*:*

1. *each $F \in \mathcal{F}$ is a closed Jordan curve which passes through ω, and*
2. *for each pair of points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ in $\mathbb{R}^2$ there exists a function $F_{p_1 p_2} \in \mathcal{F}$ whose graph passes through p_1 and p_2.*

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[1] Here, ω denotes the point at infinity.

The curves in $\mathcal{F}$ are the 1-affines of the geometry.

If $a, b, c, d \in \mathbb{R}$, $a < b$ and $c < d$, the Drandell geometry can also be defined on the bounded open rectangle $(a, b) \times (c, d)$ through a simple contraction of $\mathbb{R}^2$.

One can verify that the Drandell geometry satisfies all ten axioms in [2] and, being defined on $\mathbb{R}^2$, is a complete ordered incidence geometry (COIG). Most of the results necessary for this task are, in one form or another, a consequence of results proved in [3]. Also, whenever no confusion arises, we denote the Drandell geometry by $\mathcal{G}_{\mathcal{F}}$.

Finally, under any of these geometries, convexity can be defined, using the notation from [2], thus:

Definition 4 *Let X be a space with geometry $\mathcal{G}_{\mathcal{F}}$. A set $S \subset X$ is* convex *if for any two points $p, q \in S$, the open segment (p,q) (of the $\mathcal{G}_{\mathcal{F}}$-line $F_{p,q}$) is contained in S. The* convex hull *of any set $S \subset X$, Hull(S), is the intersection of all convex sets containing S.*

3 An extension of Kirkpatrick and Seidel's convex hull algorithm

To extend a convex hull algorithm to a geometry $\mathcal{G}_{\mathcal{F}}$, we need to be able to compute predicates and constructions that are done in the Euclidean plane using vector space operations. We have at our disposal the axioms of COIG [2] (and their consequences), which give us also an order relation (betweeness), but we do not have the parallel axiom.

Let S be a set n of points on some geometry $\mathcal{G}_{\mathcal{F}}$.

The reader familiar with Kirkpatrick and Seidel's algorithm will perceive that we need to be able to:

I. determine extremal points of the "upper" and "lower" (in some convenient sense) hull chains;

II. find a $\mathcal{G}_{\mathcal{F}}$-line F dividing the set S in two halves in such a way that F intersects an edge (bridge) of the upper hull chain;

III. settle on elimination criteria for determining the endpoints of that bridge.

We now sketch how I., II. and III. can be effected.

In the case of the Beckenbach geometry, the presence of vertical (straight) lines allow I. and II. to be easily done: just work with the abscissae of the points.

For the Drandell geometry, I. and II. can be achieved as follows. Using the betweeness relation, we have been able to show that it is possible to determine a point p_0 outside of Hull(S). Drandell [3] proved that through such a point it is possible to determine a pair of supporting $\mathcal{G}_{\mathcal{F}}$-lines for the set S, hence accomplishing I.

This pair of lines define a *cone* $\mathcal{C}$ within which the relative order of the points of S with respect to p_0 can be established. We say that a point $q \in S$ precedes another point $p \in S$ with respect to p_0 whenever the $\mathcal{G}_{\mathcal{F}}$-line $F_{p_0 q}$ precedes the $\mathcal{G}_{\mathcal{F}}$-line $F_{p_0 p}$ in clockwise order around p_0 (starting at the leftmost boundary of the $\mathcal{C}$). Based on this order, II. can also be done.

As for III., the following approach works for both types of geometries. We need to develop a predicate that will allow us to eliminate candidates to endpoints of the bridge. In the Euclidean case, this is attained by comparing the slopes of supporting lines. In the absence of the slope, we devise the following predicate for this purpose.

Given two $\mathcal{G}_{\mathcal{F}}$-lines F_1 and F_2 that intersect within the cone $\mathcal{C}$, let p be their intersection point. We say that, to the right of p, F_1 is below F_2 whenever F_1 precedes F_2 in counterclockwise order around a (sufficiently small) ball B centered at p, starting at either of the intersection points of B and $F_{p_0 p}$.

As this completes the description of the necessary predicates and constructions, a word on complexity. Since no step of the original algorithm was modified and none was added, the only possible changes to its complexity would have to be due to the new operations. However, it can be shown that except for a factor which depends only on the complexity of the $\mathcal{G}_{\mathcal{F}}$-lines, each of the reformulated predicates and construction steps takes constant time. Therefore, the algorithm still has time complexity of $O(n \log h)$, where h is the number of vertices of Hull(S).

4 Conclusion

We have presented the Beckenbach and Drandell geometries in the context of COIGs [2] and have described how to substitute the Euclidean version of predicates and constructions by those with equivalent effect on these geometries. In doing so, we provided all that is necessary to extend Kirkpatrick and Seidel's convex hull algorithm.

Similar extensions are certainly possible for other algorithms.

An expanded version of this short paper is forthcoming, in which we further extend these results to complete ordered incidence geometries in general.

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