

Extremal problems on triangle areas in the plane and in three-space

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1 Introduction

Extremal problems on triangle areas have been first investigated by Erdős and Purdy in the early 1970s [10, 11, 12]. Here we make a new round on some old and some new variants of triangle area problems in two and three dimensions.

In the plane we prove: (i) The number of triangles of minimum (nonzero) area determined by n points is at most $n^2 - n$; this is (roughly) within a factor of 2 of the current best lower bound. (ii) The number of acute triangles of minimum area determined by n points is $O(n)$; this is asymptotically tight. (iii) For points in convex position, the number of triangles of minimum area is $O(n)$; this is asymptotically tight. (iv) For points in general position (no three collinear), there exist n -element point sets that span $\Omega(n \log n)$ triangles of minimum area.

In three-space we prove: (i) The number of minimum area triangles is $O(n^2)$; this is asymptotically tight. (ii) There exist n -element point sets that span $\Omega(n^{4/3})$ triangles of maximum area. (iii) The number of unit area triangles spanned by n points is at most $O(n^{5/2} \beta(n))$, where $\beta(n)$ is an extremely slowly growing function of n . (iv) A set of n points, not all on a line, determines at least $\Omega(n^{2/3} / \beta(n))$ triangles of distinct areas that share a common side (here $\beta(n)$ is an extremely slowly growing function of n).

2 Triangles of minimum area in the plane

The general case. Braß, Rote, and Swanepoel [4] have given an $O(n^2)$ upper bound on the number of minimum area triangles determined by n points, and pointed out two examples which attain this bound: (i) a $\sqrt{n} \times \sqrt{n}$ square lattice section, and (ii) two groups of $n/2$ equidistant points on two parallel lines. The latter example has about $n^2/2$ minimum area triangles. Here we employ a new very simple but effective charging scheme which assigns each triangle of minimum area to one of its longest sides. At most two triangles get assigned to each of the $\binom{n}{2}$ segments, and this yields:

Theorem 1 *The number of triangles of minimum (nonzero) area determined by n points in the plane is at most $n^2 - n$.*

Acute triangles. In the above-mentioned example of n points placed on two parallel lines there are only $n - 2$ acute triangles of minimum area, and based on this, one may think that the maximum number of acute triangles of minimum area is only linear. Indeed, the example was no exception:

Theorem 2 *The number of acute triangles of minimum area determined by n points in the plane is $O(n)$.*

Convex position. For points in convex position we prove an $O(n)$ upper bound, which is asymptotically tight (a regular n -gon has n triangles of minimum area).

Theorem 3 *The maximum number of triangles of minimum area determined by n points in convex position in the plane is $O(n)$.*

General position. For points in *general position* (i.e., no three collinear), we believe that the maximum number of triangles of minimum area is close to linear: we can prove an $\Omega(n \log n)$ lower bound, but no subquadratic upper bound for now. For the lower bound: a recent result in [8] shows that there exist n -point sets in general position with $\Omega(n \log n)$ empty congruent triangles. We only have to ensure that the construction can be carried out so that these triangles also satisfy the minimum area condition, which we do and thereby obtain:

Theorem 4 *For all n , there exist n -element point sets in general position in the plane that span $\Omega(n \log n)$ triangles of minimum area.*

3 Triangles in the space

In the sequel, we use $\beta(n)$ to denote any function of the form $\exp(\alpha(n)^{O(1)})$, where $\alpha(n)$ is the extremely slowly growing inverse Ackermann function. Any function $\beta(n)$ is also extremely slowly growing.

Minimum area. Similarly to the planar example, one can place n equally spaced points on the three parallel edges of a prism, whose base is an equilateral triangle, such that inter-point distances are small on each edge. This yields about $2n^2/3$ minimum area triangles. The

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following theorem shows that this bound is optimal up to a constant factor. The upper bound is based on the same charging scheme that we use in the planar case, which assigns each triangle of minimum area to one of its longest sides. It turns out that at most a constant number of triangles get assigned to each of the $\binom{n}{2}$ segments, which implies:

Theorem 5 *The number of minimum area triangles in \mathbb{R}^3 is $O(n^2)$.*

Maximum area. Ábrego and Fernández-Merchant [1] showed that one can place n points on the unit sphere in \mathbb{R}^3 so that they determine $\Omega(n^{4/3})$ pairwise distances of $\sqrt{2}$ (see also p. 261 in [3]). They also showed that one can obtain the same number $\Omega(n^{4/3})$ of congruent copies of any given triangle T . By selecting T as a right angled isosceles triangle, we can also ensure that T has maximum area among all triangles, and thus obtain the following result:

Theorem 6 *There exist n -element point sets in \mathbb{R}^3 that span $\Omega(n^{4/3})$ triangles of maximum area.*

Unit areas. Erdős and Purdy [10] showed that a suitable (very thin) section of the integer lattice containing n points determines $\Omega(n^2 \log \log n)$ triangles of the same area. Clearly, this bound is also valid in three-space. An upper bound of $O(n^{8/3})$ on the number of unit triangle areas in 3D has been also derived in the same paper.

Using a decomposition method of Chazelle et al. [6], and following the partition technique of Clarkson et al. [7], we prove an upper bound on the number of point-cylinder incidences in \mathbb{R}^3 , when the axes of all cylinders pass through the origin.

Lemma 7 *Given a set S of n points and a set D of m (circular) cylinders in three-space with their axes passing through the origin, the number of point-cylinder incidences is bounded by*

$$I_R(S, D) = O(n^{3/4} m^{3/4} \beta(n) + n\beta(n) + m).$$

Using Lemma 7, we can further show that given a set S of n points in three-space and any point $p \in S$, the number of unit area triangles spanned by S and incident to p is at most $O(n^{3/2} \beta(n))$. By summing over all points in S we obtain an improvement over the old $O(n^{8/3})$ bound:

Theorem 8 *The number of unit area triangles spanned by n points in three-space is at most $O(n^{5/2} \beta(n))$.*

Distinct areas. We show that every set S of n points in three-space determines almost $\Omega(n^{2/3})$ triangles with distinct areas that share a common side. For triangle areas in the plane, we have a similar phenomenon. By an argument of Burton and Purdy [5], every set S of n

non-collinear points in the plane determines $\Omega(n)$ triangles with distinct areas which share a common side. The current record is $\frac{17}{38}n - O(1) \approx 0.4473n$ [9]. The tools we use in obtaining the 3D lower bound are a famous result of Beck [2] and Lemma 7 applied to bound incidences between the points and a suitable set of cylinders.

Theorem 9 *A set of n points in \mathbb{R}^3 , not all on a line, determines at least $\Omega(n^{2/3} / \beta(n))$ triangles of distinct areas. Moreover, all these triangles share a common side.*

References

- [1] B. M. Ábrego and S. Fernández-Merchant, Convex polyhedra in \mathbb{R}^3 spanning $\Omega(n^{4/3})$ congruent triangles, *J. Combin. Theory, Ser. A* **98** (2002), 406–409.
- [2] J. Beck, On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry, *Combinatorica* **3** (1983), 281–297.
- [3] P. Braß, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer, New York, 2005.
- [4] P. Braß, G. Rote, and K. J. Swanepoel, Triangles of extremal area or perimeter in a finite planar point set, *Discrete Comput. Geom.* **26** (2001), 51–58.
- [5] G. R. Burton and G. Purdy, The directions determined by n points in the plane, *J. London Math. Soc.* **20** (1979), 109–114.
- [6] B. Chazelle, H. Edelsbrunner, L. J. Guibas, and M. Sharir, A singly-exponential stratification scheme for real semi-algebraic varieties and its applications, in *Proc. 16th ICALP*, vol. 372 of LNCS, Springer, Berlin, 1989, pp. 179–193.
- [7] K. L. Clarkson, H. Edelsbrunner, L. G. Guibas, M. Sharir, and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.* **5** (1990), 99–160.
- [8] A. Dumitrescu, J. Pach, and G. Tóth, The maximum number of empty congruent triangles determined by a point set, *Revue Roumaine de Math. Pures et Appliquées* **50** (2005), 613–618.
- [9] A. Dumitrescu and C. D. Tóth, Distinct triangle areas in a planar point set, manuscript 2006.
- [10] P. Erdős and G. Purdy, Some extremal problems in geometry, *J. Combinatorial Theory* **10** (1971), 246–252.
- [11] P. Erdős and G. Purdy, Some extremal problems in geometry IV, *Congressus Numerantium* **17** (Proc. 7th South-Eastern Conf. on Combinatorics, Graph Theory, and Computing), 1976, 307–322.
- [12] P. Erdős and G. Purdy, Some extremal problems in geometry V, *Proc. 8th South-Eastern Conf. on Combinatorics, Graph Theory, and Computing*, 1977, 569–578.