Optimal Linear Time Algorithm for Intensity Map Splitting with Feathering in Radiation Therapy

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1 Introduction

In this paper, we study a geometric partition problem, called *field splitting using vertical lines with feathering*, which arises in *Intensity-Modulated Radiation Therapy* (IMRT) [14], a state-of-art radiation therapy technique for cancer treatments, aiming to deliver a highly conformal radiation dose to a target tumor while sparing the surrounding normal tissues. The prescribed dose distribution of radiation is commonly specified by a set of nonnegative integers on a 2-D grid showing the amount of radiation to be delivered to the body region corresponding to the cells, which is called *intensity map* (IM).

One of the most advanced tools today for delivering IMs is the *multileaf collimator* (MLC)[14]. An MLC has multiple pairs of tungsten leaves of the same rectangular shape and size aligned to each other. The leaves can move up and down to form a rectilinear region, called *MLC-aperture*. Radiation beam is shaped by this MLC-aperture to deliver certain units of radiation to an IM.

The limited size of the MLC necessitates that a large-size IM field be split into two or more adjacent subfields, each of which can be delivered separately by the MLC [6, 7, 15]. But, such IM splitting may result in prolonged treatment time and increased deliver error, and thus affect the treatment quality. The field splitting problem is to split an IM of a large size into multiple sub-IMs whose sizes are all no bigger than a threshold size, such that the treatment quality is optimized.

A few field splitting algorithms are known in the literature[5, 11, 10, 16], which address various versions of the field splitting problem. However, all these field splitting solutions focused on minimizing the beam-on time while ignoring the issue of reducing the delivery error. For an IM M of size $m \times n$, the minimum amount of error for delivering M is captured by the formula[3]: $Err(M) = \sum_{i=1}^{m}(M_{i,1} + \sum_{j=1}^{n-1} |M_{i,j} - Mi, j+1| + M_{i,n})$. This delivery error is caused by the tongue and groove effect of the MLC leaves, which has been studies by several literature[4, 8, 9, 12] but not for splitting.

Geometrically, if we view a row of an IM as representing an x-monotone rectilinear curve f, called the *dose profile curve*, then the delivery error associated with this row is actually the total sum of the lengths of all vertical edges on f (Figure 1(a)).

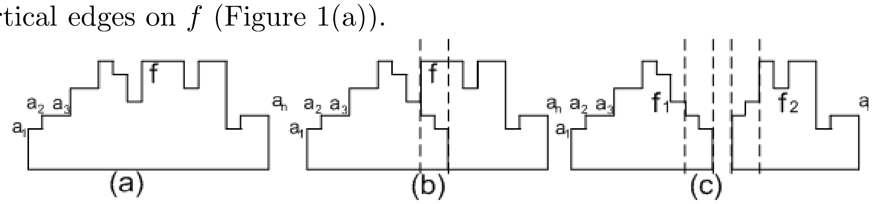

Figure 1: (a) The dose profile curve f of one row of an IM. The delivery error, $Err(f)$, of the row is equal to the sum of the lengths of all vertical edges on curve f. (b) Splitting the one-row IM in (a) into two sub-IMs with feathering. (c) The two resulting sub-IMs.

In this paper, we consider the following **field splitting using vertical lines with feathering problem**: Given an IM M of size $m \times n$, a maximum field width w and the width of each feathering region δ, with $n > w$, split M using vertical lines into $d = \lceil \frac{n-\delta}{w-\delta} \rceil$ (≥ 2) sub-IMs $M_1, M_2, \ldots, M_d$, each with a width no larger than w, such that the total delivery error of these d sub-IMs is minimized. Only adjacent two sub-IMs overlap with each other and the width of the feathering (overlapping) region is δ. Note that d is the minimum number of sub-IMs needed for delivering M. We may use more sub-IMs. But, that could significant increase the total treatment time, which is undesirable.

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In our approach, we model the problem as computing a shortest path in a directed acyclic graph (DAG) with $O(n)$ vertices and $O(wn)$ edges. The computation of each edge weight takes pseudo-polynomial time. Interestingly, we are able to calculate each edge weight in constant time after a certain preprocessing. Moreover, the edge weights satisfy the Monge property[1, 2, 13]. Thus, we can solve this shortest path problem by examining only a very small portion of the edges of the graph, and our algorithm runs in an optimal $O(mn)$ time.

2 Overview of the Algorithm

Since the width of each feathering region is fixed as δ, d vertical lines are needed to determine the split (including the last vertical line which is always corresponding to the last column). We denote these vertical lines as $c_1, c_2, \ldots, c_d$, where c_j is the number of the column that the j-th vertical line cut through the IM immediately after. The j-th feathering region consists of δ columns starting from Column $c_j - \delta + 1$, which is denoted by a *column-bounding interval* $I_j = [le..re]$.

Our algorithm development starts with the observation that whenever we use two vertical lines to form a feathering region between M_k and M_{k+1}, the total delivery error of the resulting sub-IMs will *increase* by a value of *at least* $2\sum_{i=1}^{m}(\sum_{j=le}^{re}|M_k[i,j] - M_k[i,j-1]| + M_k[i,re])$ (Figure 1), which is also called the cost of the feathering region. Furthermore, the cost of the feathering region can be achieved by an optimal decomposition of the feathering region. It can be shown that for a given feathering region $M[le,\ldots,re]$, if we define the sequence $\{x_l\}_{l=le}^{re}$ by $x_{re} = M_k[i,re]$ and the recursive formula $x_{l-1} = \max\{0, x_k - \max(0, M_k[i,l] - M_k[i,l-1])\}$, the split $\{M_k[i,j] = x_j : j = le,\ldots,re\}$ and $\{M_{k+1}[i,j] = M[i,j] - x_j \ : \ j = le,\ldots,re\}$ is actually an optimal split on the given feathering region with minimum increase of the delivery error. It is clear that by pre-calculating the sum $E_j = \sum_{i=1}^{m}\sum_{l=1}^{j}|M[i,l] - M[i,l-1]|$ for each $j = 1, 2, \ldots, n$ (which totally takes $O(mn)$ time), the cost of the feathering region can be computed in $O(1)$ time.

We next construct the DAG $G = (V, E)$ used for the search of the optimal split of M. First, we exploit the possible positions of each vertical line. Let $\mu = (n-\delta) \bmod (w-\delta)$. If $\mu = 0$, set $\mu = w - \delta$. Notice that for the j-th ($j \neq d$) vertical line, there are j sub-IMs to the left and $d-j$ sub-IMs (including the one overlapping with it) to the right of it. Then the possible position set C_j of the j-th vertical line is $C_j = \{l : j(w-\delta) + 2\delta + \mu - w \leq l \leq j(w-\delta) + \delta\}$. We now can define the graph G, which has $d+1$ layers of vertices. The first layer only has one dummy vertex $V_{0,0}$ and the last layer has one vertex $V_{d,0}$ corresponding to the last vertical line $c_d = n$. There are $d-1$ layers of vertices in between, each of which consists of $|C_j|$ vertices $\{V_{j,l} : j(w-\delta) + 2\delta + \mu - w \leq l \leq j(w-\delta) + \delta\}$, where $V_{j,l}$ corresponds to a possible vertical line position $c_j = l$. An edge is then put in E from $V_{j,l}$ in Layer j to every $V_{j+1,l'}$ in Layer $j+1$ ($j = 1, 2, \ldots, d$) if the distance between the two corresponding vertical lines is no larger than w, and the weight of the edge is the cost of its corresponding feathering region of $V_{j,l}$. We can prove that each $V_{0,0}$-to-$V_{d,0}$ path specifies a feasible split of M and vice verse. Furthermore, the weight of the path equals to the (increased) value of the delivery error of the corresponding split. Our algorithm then computes a shortest path from $V_{0,0}$-to-$V_{d,0}$ in G, which defines an optimal split of M. It is easy to see that G is a DAG with $O(n)$ vertices and $O(dw^2) = O(n^2/d)$ weighted edges. Thus, a shortest $V_{0,0}$-to-$V_{d,0}$ path can be computed in $O(n^2/d + mn)$ time, where $O(mn)$ is the time used to compute the edge weights.

The following Lemma reveals the Monge property of the edge weights in G, leading to an optimal linear time algorithm for solving the field splitting problem.

Lemma 1 *Let V_{j,l_j}, V_{j,l'_j}, $V_{j+1,l_{j+1}}$, and $V_{j+1,l'_{j+1}}$ be four vertices in two adjacent layers of G, with $l_j < l'_j$ and $l_{j+1} < l'_{j+1}$. If $e_1 = (V_{j,l_j}, V_{j+1,l'_{j+1}})$ and $e_2 = (V_{j,l'_j}, V_{j+1,l_{j+1}})$ are both edges of G, then $e_3 = (V_{j,l_j}, V_{j+1,l_{j+1}})$ and $e_4 = V_{j,l'_j}, V_{j+1,l'_{j+1}}$ are both edges of G. Moreover, $w(e_3) + w(e_4) \leq w(e_1) + w(e_2)$.*

The Monge property stated in Lemma 1 indicates that we can always assume that no two shortest $V_{0,0}$-to-$V_{d,0}$ paths "cross" each other. Given the shortest paths from $V_{0,0}$ to all vertices on Layer j, by using this "non-crossing" property of the shortest paths, only $O(w)$ edges need to be examined in order to find the shortest path from $V_{0,0}$ to every vertex on Layer $j+1$. Thus, only $O(dw) = O(n)$ edges are examined to compute the shortest $V_{0,0}$-to-$V_{d,0}$ path in G. Hence, the total time complexity for solving the field splitting problem is $O(mn)$.

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