Farthest Segment Spanned by Points in $\mathbb{R}^3$

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1 Introduction

In its general form, the problem of finding the farthest segment spanned by points is defined as follows: Given a set $S = \{p_1, p_2, \ldots, p_n\}$ of n points in $\mathbb{R}^d$, and a query point $q \in \mathbb{R}^d$, find the farthest line segment from q among the set E of $O(n^2)$ line segments spanned by S.

The problem, in its 2-dimensional version, was introduced in [2] and has sparked the development of fundamental data structures [1,4] that surprisingly enough were not addressed by previous work. In [2], they give an optimal (in the algebraic decision tree model), $O(n \log n)$ time, $O(n)$ space algorithm for solving the problem. In [4], they address the related all-farthest-segments problem, that asks to compute the farthest line segment for each of the points in S (i.e., there are n query points, each of them being a point in S), and give an optimal, $O(n \log n)$ time, $O(n)$ space algorithm, improving their previous result on the same problem [3]. A byproduct of their solution is the development of a data structure for the farthest-segment Voronoi diagram of a convex polygon. In particular, they prove the farthest-segment Voronoi diagram of an n-sided convex polygon has complexity $O(n)$ and can be computed in $O(n \log n)$ time. The result was later generalized by Aurenhammer, Drysdale, and Krasser [1], who show that the farthest line segment Voronoi diagram for a set of n segments in the plane (that are allowed to touch or cross) can be computed in $O(n \log n)$ time. They also note that the farthest line segment Voronoi diagram has properties different from both the closest-segment Voronoi diagram and the farthest-point Voronoi diagram.

Although not specifically stated there, the result in [4] implies that a set S of n points in the plane can be preprocessed in $O(n \log n)$ time and $O(n)$ space so that given a query point q in the plane, the farthest line segment from q, defined by the points in S, can be reported in $O(\log n)$ time.

We prove that a key property in [4] can be extended to $\mathbb{R}^3$ and give an algorithm for the 3-dimensional version of the problem that matches the time and space complexities of the planar version [2]. The algorithm is optimal in the algebraic decision tree model. If preprocessing is allowed on S, answering queries efficiently would require a data structure for the farthest-segment Voronoi diagram of an n-sided convex polytope. We leave the complexity analysis and construction of this diagram as an open problem.

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2 Farthest line segment in $\mathbb{R}^3$

For a set of points S, let $CH(S)$ denote the convex hull of S. Our algorithm makes use of:

Lemma 1 *Given a set $S = \{p_1, p_2, \ldots, p_n\}$ of n points in $\mathbb{R}^3$, and a query point $q \in \mathbb{R}^3$, the farthest line segment from q among the set E of $O(n^2)$ line segments spanned by S has at least one endpoint at a vertex of $CH(S)$. Moreover, (i) if both endpoints are vertices of $CH(S)$ then the line segment is an edge of $CH(S)$ and (ii) if only one endpoint is a vertex of $CH(S)$ then the other endpoint is the farthest point from q among those points in S that are not vertices of $CH(S)$.*

Note that if the farthest line segment p_ip_j, where $1 \leq i, j \leq n$, is in case (ii), with p_i a vertex of $CH(S)$, then p_i is the only vertex of $CH(S)$ contained in the halfspace H_j defined by a plane orthogonal to qp_j at p_j, and such that $q \notin H_j$.

Using the result in Lemma 1, we have the following simple algorithm. We start by computing the convex hull $CH(S)$ of S, which takes $O(n \log n)$ time. From the edges of $CH(S)$, select the farthest one, e_1. This segment is one of the two possible candidates for the farthest line segment, according to Lemma 1, and can be found easily in $O(n)$ time. Let V denote the set of points that are vertices of $CH(S)$ and let $S' = S \setminus V$. We next find the farthest point from q in S', which takes $O(n)$ time. Let this point be p_i, where $1 \leq i \leq n$. If p_i is closer to q than e_1 then report e_1 as the farthest line segment. Else, find the farthest line segment from q with an endpoint at p_i and the other endpoint in V, which can be done in $O(n)$ time, and report this segment, e_2, as the farthest line segment.

Theorem 1 *Given a set $S = \{p_1, p_2, \ldots, p_n\}$ of n points in $\mathbb{R}^3$, and a query point $q \in \mathbb{R}^3$, the farthest line segment from q among the set E of $O(n^2)$ line segments spanned by S can be found in $O(n \log n)$ time using $O(n)$ space.*

We can show how to extend the results above to the related query answering problem, in which S is given in advance and can be preprocessed for fast queries. However, answering queries efficiently would require a data structure for the farthest-segment Voronoi diagram of an n-sided convex polytope. We leave the complexity analysis and construction of this diagram as an open problem.

References

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