Automorphisms and Distinguishing Numbers of Geometric Cliques

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Let $\overline{G}$ denote a geometric graph. In particular, $V(\overline{G})$ is a set of points in general position in $\mathbb{R}^2$ and the edge $uv \in E(\overline{G})$ is the straight line segment joining the corresponding pair of points. Two edges, say uv and xy, are said to *cross* if the interiors of the line segments from u to v and x to y have nonempty intersection. A bijection from $V(\overline{G})$ to itself is called a *geometric automorphism* if it preserves adjacency and non-adjacency of vertices, as well as crossing and non-crossing of edges. We let $\overline{K}_n$ denote a *geometric clique* (or a geometric complete graph) on n vertices. It is convenient to denote the boundary of the convex hull of $\overline{K}_n$ by C. We begin by presenting two theorems describing constraints of the action of a geometric automorphism on C.

Theorem 1. Any geometric automorphism that fixes each vertex on the boundary of the convex hull of $\overline{K}_n$ fixes every vertex of the graph.

We prove this theorem by assuming there is an automorphism f that fixes the vertices of C, but $f(x) = y$ for two distinct vertices interior to C. If we repeatedly apply this automorphism we must get $f^r(x) = x$ for some integer r. A contradiction is realized when we see that this implies a pair of uncrossed edges had to be crossed in the process.

Theorem 2. If the boundary of the convex hull of $\overline{K}_n$ contains at least four vertices, then every geometric automorphism sends vertices on the boundary of the convex hull to vertices on the boundary of the convex hull.

If there are fewer than two vertices interior to C the proof is easy, so assume that there are at least two such vertices. Since there are at least four vertices in C, the area interior to C is partitioned by edges into (more than one) disjoint regions. If there is a geometric automorphism taking C off itself it must take all the vertices of C to a single region of this partition. A careful study shows that this requires that at least one edge in the image of C must be crossed. But this is impossible because C is a cycle of uncrossed edges and a geometric automorphism must preserve the property of being uncrossed.

Note that a $\overline{K}_6$ consisting of two nested 3-cycles illustrates that Theorem 2 cannot be extended to geometric cliques with three vertices on the boundary of the convex hull. In this case there is a geometric automorphism that swaps these 3-cycles. See Figure 1.

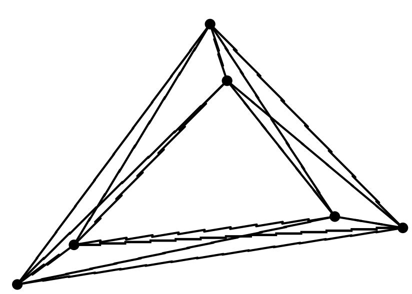

Figure 1: The boundary of the convex hull is not invariant.

This paper uses the above theorems along with results from determining sets [4] to prove a theorem involving distinguishing labelings of geometric cliques.

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A labeling $f : V(\overline{G}) \to \{1, \dots, d\}$ is said to be *d-distinguishing* if $\phi \in \mathrm{Aut}(\overline{G})$ and $f(\phi(x)) = f(x)$ for all $x \in V(\overline{G})$ implies that $\phi = id$. Note that we have a d-distinguishing labeling of a geometric graph $\overline{G}$ if and only if the labeling together with the structure of $\overline{G}$ uniquely identifies every vertex. Every geometric graph has a distinguishing labeling since we can label each vertex with a different integer from $\{1, \cdots, |V(\overline{G})|\}$. Distinguishing labelings were introduced in [3] and have seen substantial recent interest. In previous work [1], the authors proved that every geometric clique (except the non-convex $\overline{K}_4$) can be 3-distinguished. Here we prove that when $n \geq 7$ and the number of vertices in C is at least four, $\overline{K}_n$ can be 2-distinguished.

A subset $S \subseteq V(\overline{G})$ is said to be a *determining set* if whenever g, h are geometric automorphisms of G and $g(s) = h(s)$ for all $s \in S$, then $g = h$. Every graph has a determining set, since any set containing all but one vertex is determining. The determining set was introduced in [4], was first used for distinguishing in [2], and can be characterized as a set in which the only automorphism that fixes all its vertices is the trivial automorphism. Theorem 1 then tells us that the set of vertices of C, the boundary of the convex hull of the geometric clique, is a determining set. Further, Theorem 2 can be used with results from [4] to show that any pair of non-antipodal vertices of C is a determining set for $\overline{K}_n$.

We will also need to define what it means to d-distinguish a subset of $V(\overline{G})$. Let $S \subseteq V(\overline{G})$. A labeling $f : S \to \{1, \cdots, d\}$ is called *d-distinguishing* if $\phi \in \mathrm{Aut}(\overline{G})$ and $f(\phi(x)) = f(x)$ for all $x \in S$ implies that ϕ fixes the vertices of S. Note that such a ϕ may not be the identity (it might move vertices that are not in S), but it fixes S pointwise. This notion of distinguishing a subset enables us to connect determining sets and distinguishing labelings.

Theorem 3. [2] $\overline{G}$ can be d-distinguished if and only if $V(\overline{G})$ contains a determining set that can be $(d-1)$-distinguished.

We use Theorem 3 to prove the following.

Theorem 4. If $n \geq 7$ and $\overline{K}_n$ has at least four vertices on the boundary of its convex hull then $\overline{K}_n$ can be 2-distinguished.

The proof is easy if there are fewer than two vertices interior to C. Suppose there are at least two vertices, x and y, interior to C. We find vertices u_1, u_2, u_3 in C so that u_1, u_1, u_3, x form a convex quadrilateral with y in its interior. Let $\overline{H}$ be the subgraph induced by $S = \{u_1, u_2, u_3, x, y\}$. We first show that there is no geometric automorphism permuting u_1, u_2 and u_3. Further, since there is no geometric automorphism of $\overline{K}_n$ moving x to any u_i (by Theorem 2 applied to $\overline{K}_n$) and there is no geometric automorphism of $\overline{H}$ moving y to any of $\{u_1, u_2, u_3, x\}$ (by Theorem 2 applied to $\overline{H}$) there is no nontrivial automorphism of $\overline{K}_n$ that moves any vertex of S. Thus we can 1-distinguish the vertices of S. Also note that since S contains three vertices from C, it contains two non-antipodal vertices of C and is therefore a determining set for $\overline{K}_n$. Thus 1-distinguishing S is sufficient to 2-distinguish $\overline{K}_n$.

References

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