

Planar Minimally Rigid Graphs and Pseudotriangulations

November 21, 2003

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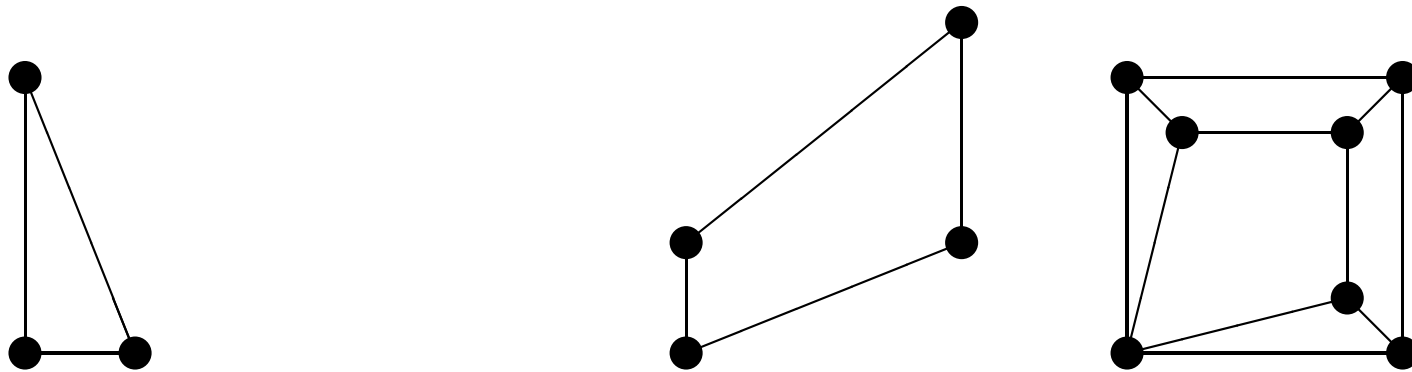
I. Rigid Graphs.

II. Robot Arms

III. Pseudotriangles

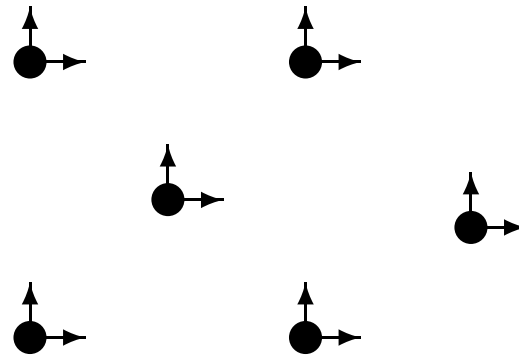
IV. Relationships

Part I: Rigidity of Graphs



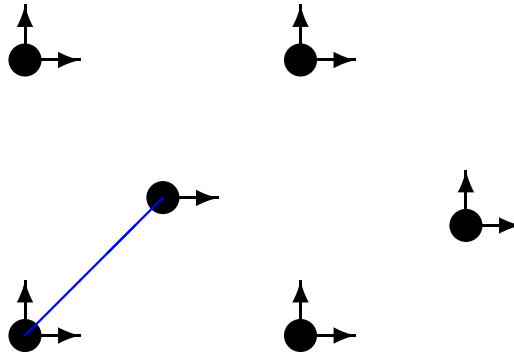
Construct the graph G in \mathbb{R}^2 using **inflexible bars** for the edges and rotatable joints for the vertices. This immersion is **rigid** if the only motions of this structure are trivial (rotations and translations).

Intuition: How many edges are necessary to make a rigid body with V vertices?



Total degrees of freedom: $2V$.

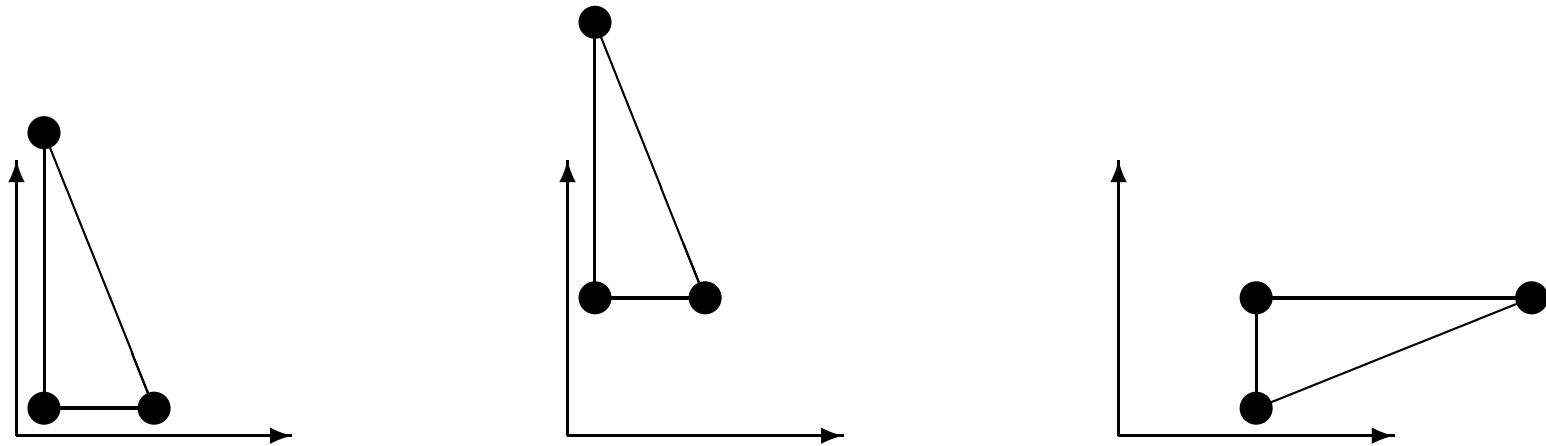
Intuition: How many edges are necessary to make a rigid body with V vertices?



Total degrees of freedom: $2V$.

Adding an edge can remove a degree of freedom.

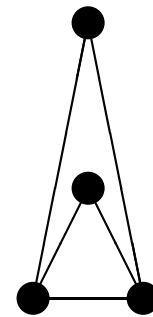
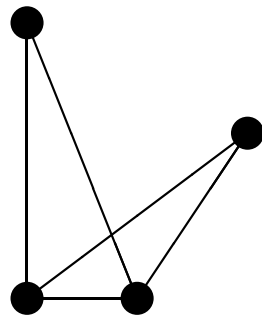
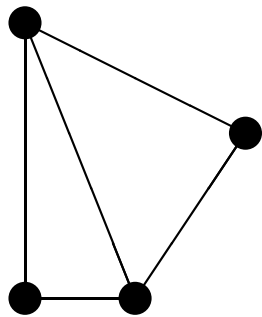
In the end you will still be able to **translate and rotate**.



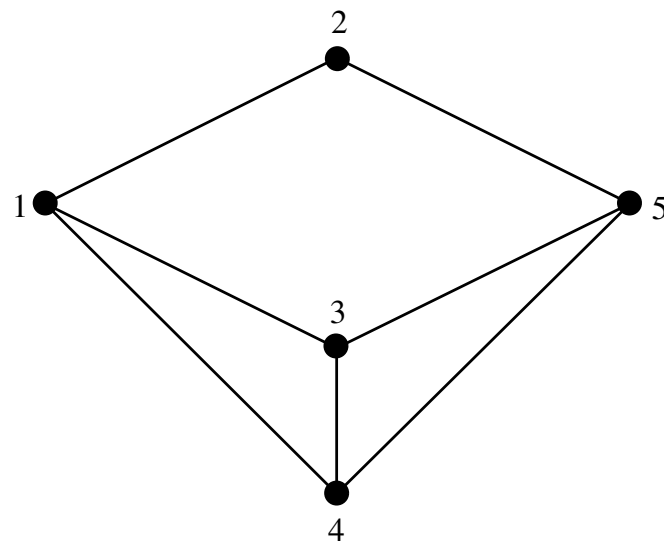
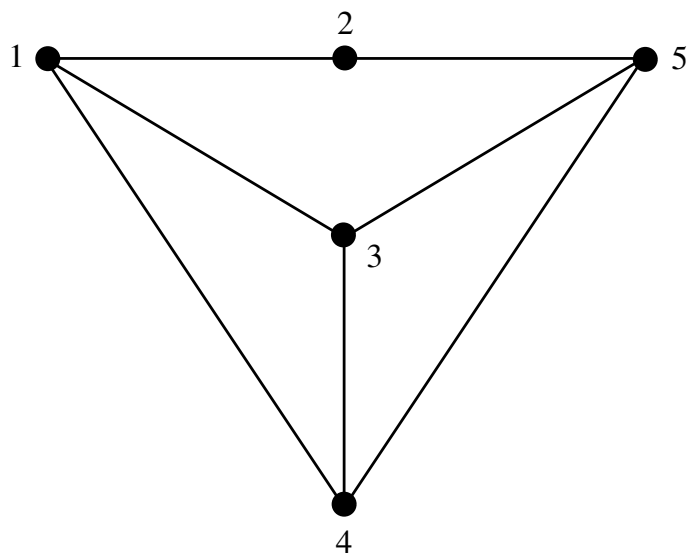
Thus, if G is rigid in \mathbb{R}^2 then $E \geq 2V - 3$.

A graph is *rigid* in \mathbb{R}^2 if some embedding of it is.

Turns out, if some embedding is rigid then most are.

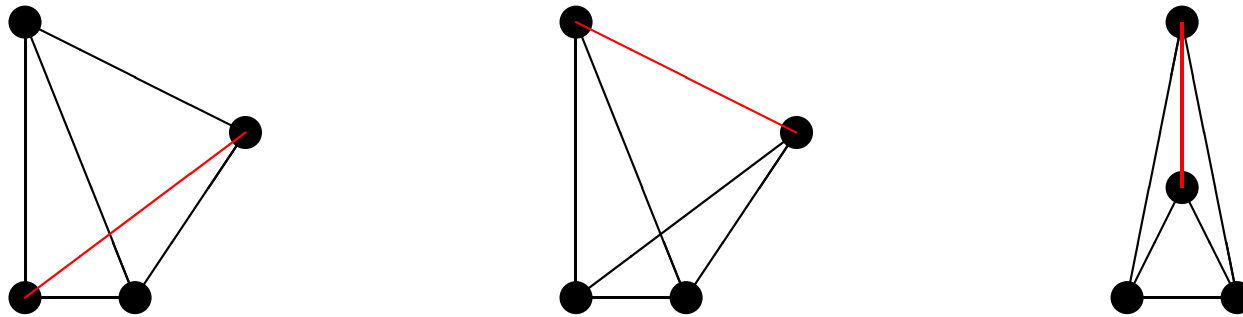


Aside: When the vertices are **not in general position**, can get non-rigid embedding. (rigid but not infinitesimally rigid)



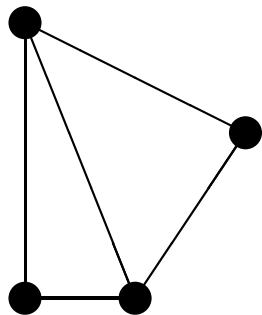
We'll be concerned with vertices in general position.
(Generic Rigidity)

A graph is minimally rigid (2-isostatic) in \mathbb{R}^2 if removing any edge results in a graph that is not rigid.



These are not minimally rigid.

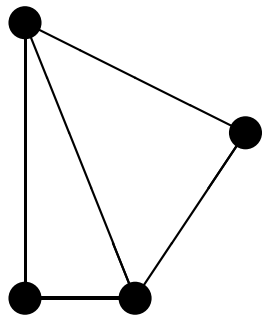
Theorem (Laman, 1970) A graph is **minimally rigid** if and only if $E_G = 2V_G - 3$ and for any subgraph H with at least 2 vertices, $E_H \leq 2V_H - 3$.



$$V = 4 \text{ then}$$

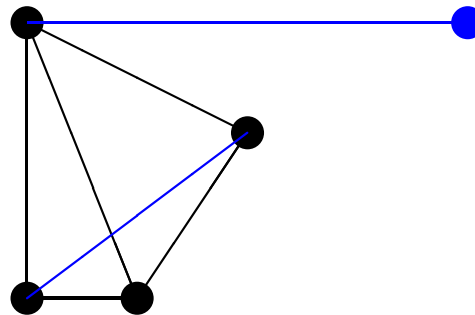
$$E = 2 * 4 - 3 = 5$$

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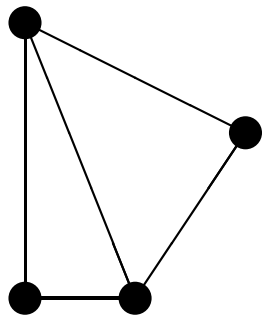
$$E = 2 * 4 - 3 = 5$$



$V = 5$ then

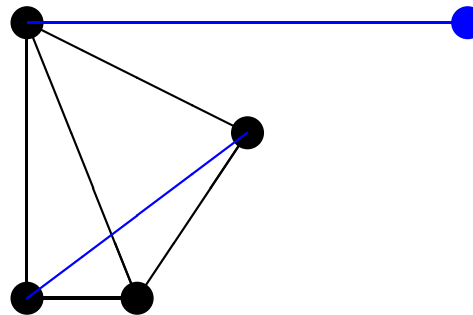
$$E = 2 * 5 - 3 = 7$$

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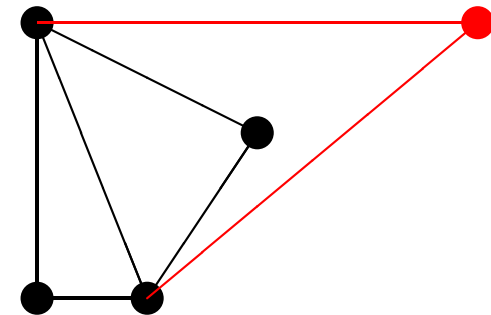
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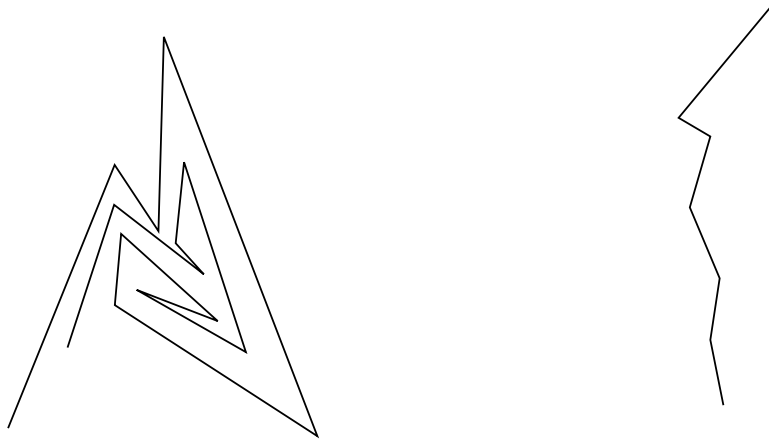


Current Applications of Rigidity.

- robotics (Streinu, Connelly, Demaine, Rote)
- geometrical properties of molecular, conformations including protein folding (Whitelely, Thorpe)
- molecular modelling, (Bezdek, Streinu, and others)

Part II: Robot arms and Carpenter's Rule

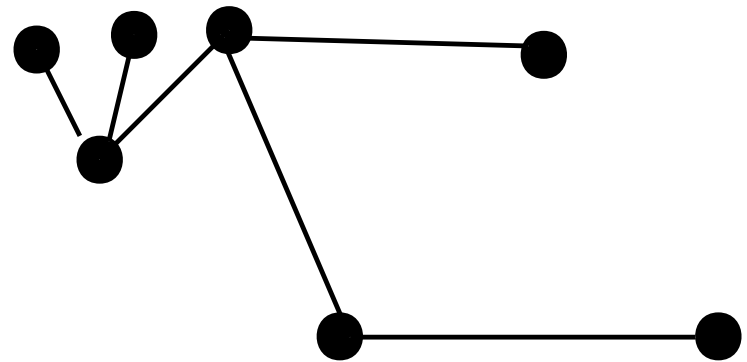
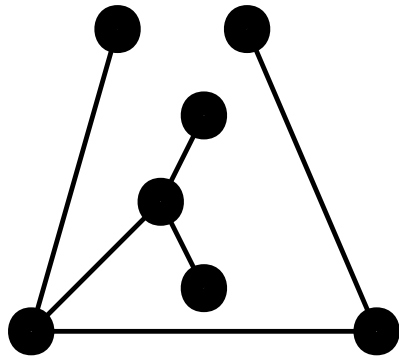
Given any planar embedding of a path can it be straightened?

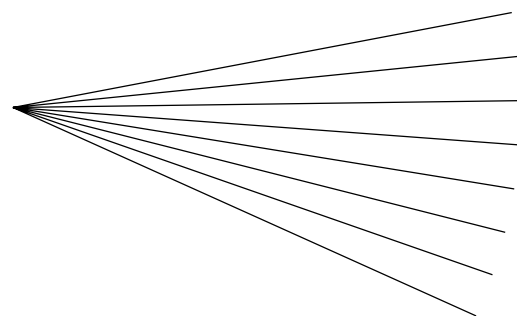
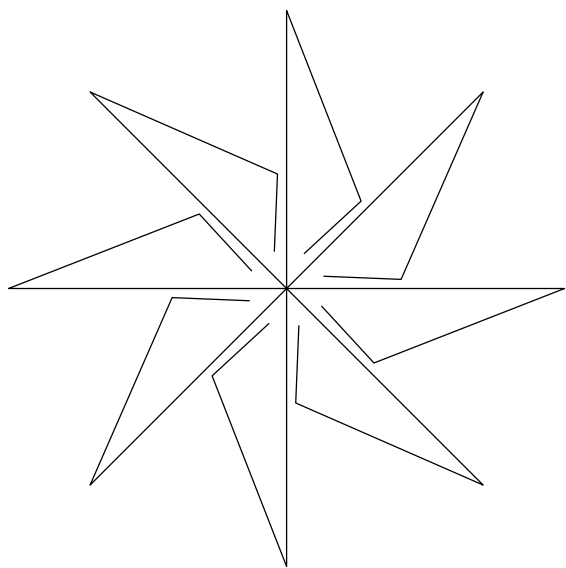


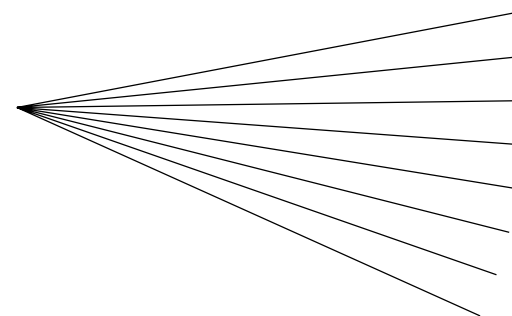
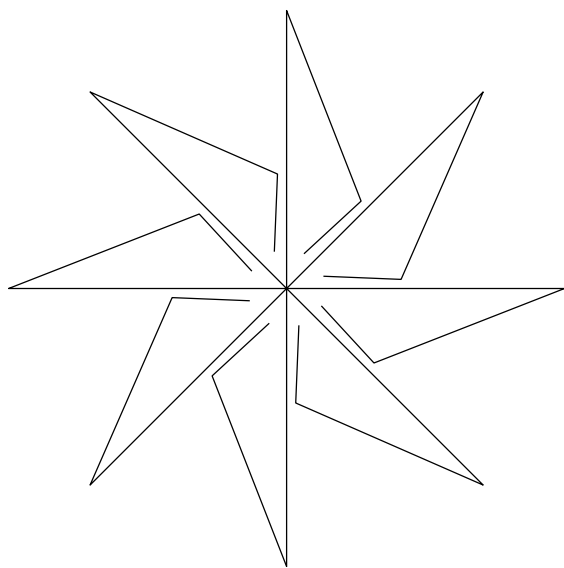
Move edges without crossing them!

What about moving one embedding of a graph to another?

Trees?



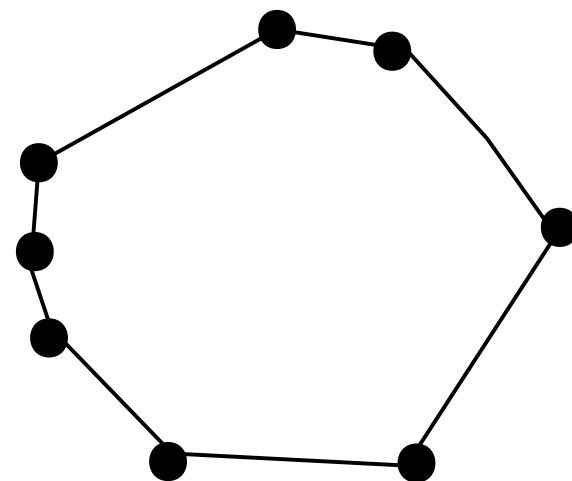
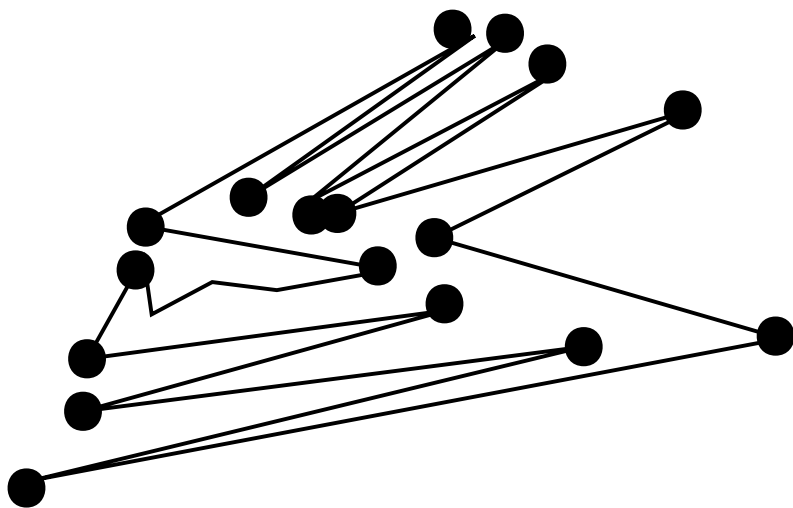




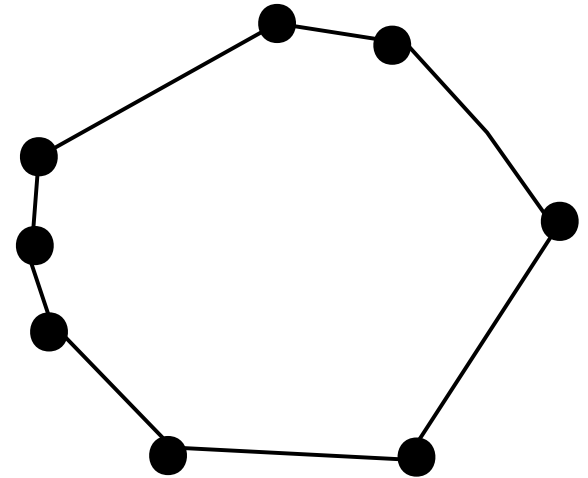
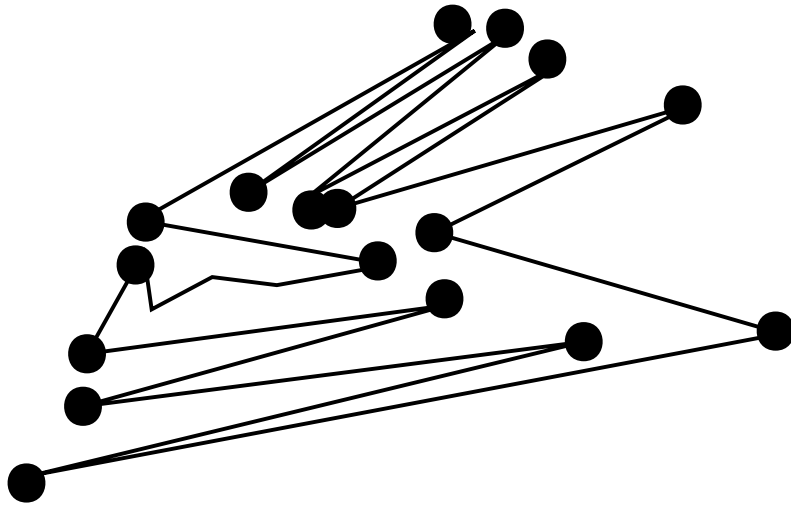
Trees can Lock!

(Beidl, Demaine, Demaine, Lubiw, O'Rourke, Overmans,
Robbins, Streinu)

Can any planar embedding of C_n be convexified?



Can any planar embedding of C_n be convexified?



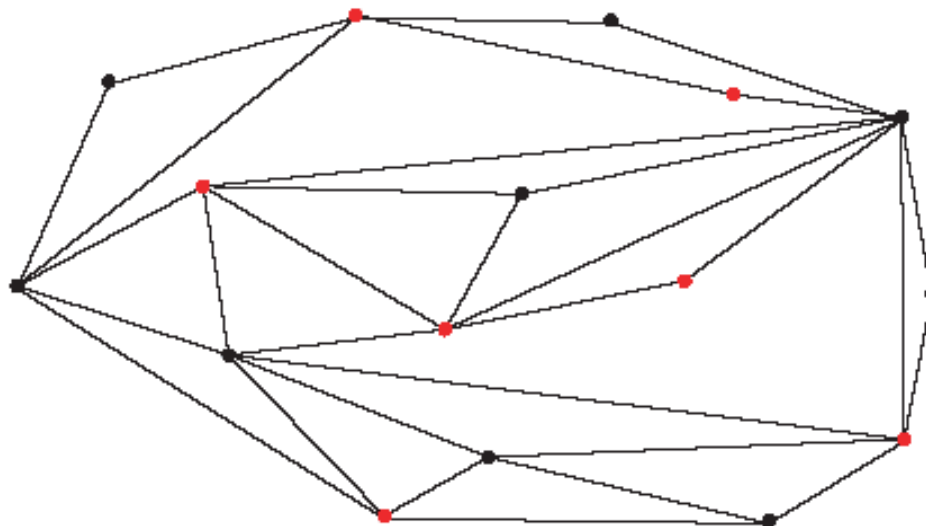
Yes! Connelly, Demaine, Rote; nicer algorithm by Streinu.

Part III: Pseudotriangles

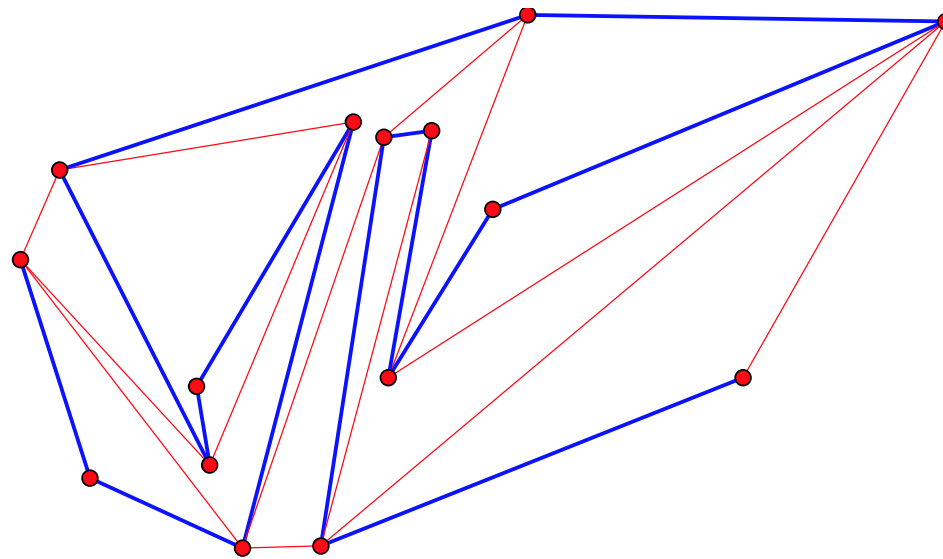
A **pseudotriangle** has 3 convex corners:



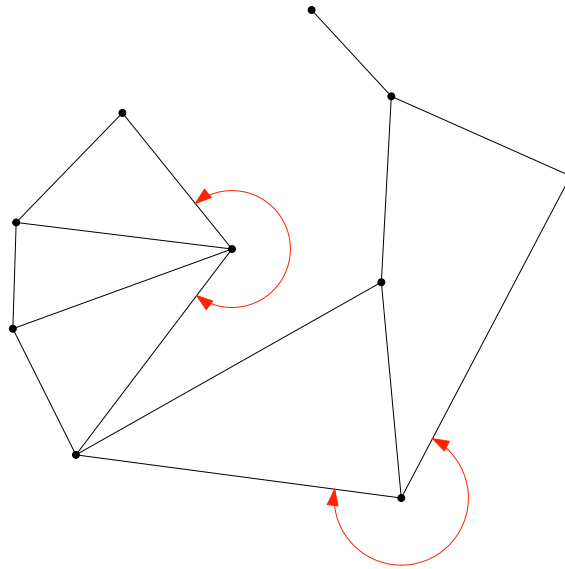
A [pseudotriangulation](#) is an embedding of a planar graph in which all interior faces are pseudotriangles. The exterior face is convex.



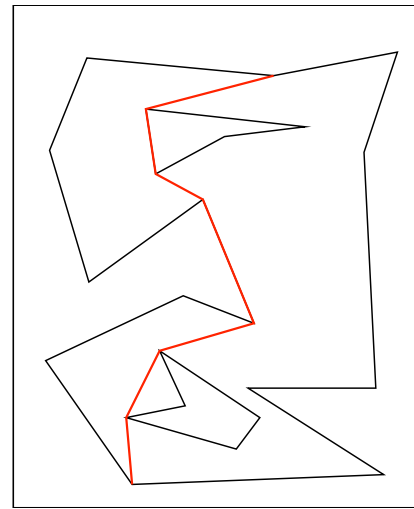
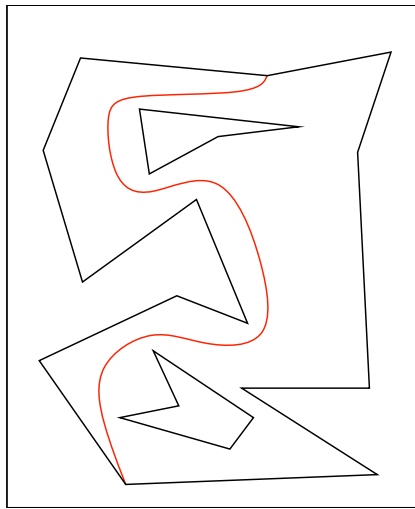
In *minimum* pseudotriangulation every vertex will be pointed.



A *pointed* vertex is incident to an angle $> 180^\circ$ (a *reflex* angle or *big* angle).



Shortest path (with given homotopy) turns only at pointed vertices. Addition of shortest path edges leaves intermediate vertices pointed.

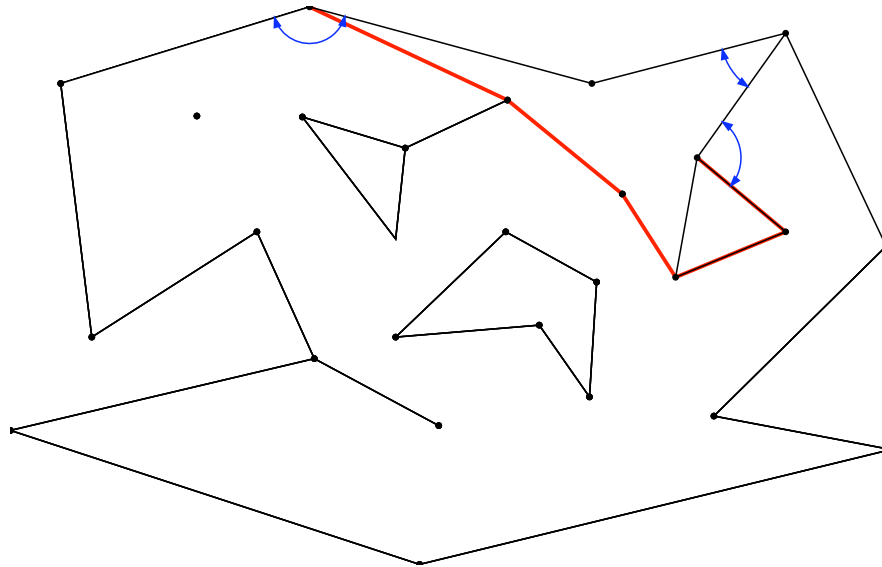


→ *geodesic* triangulations of a simple polygon

[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]

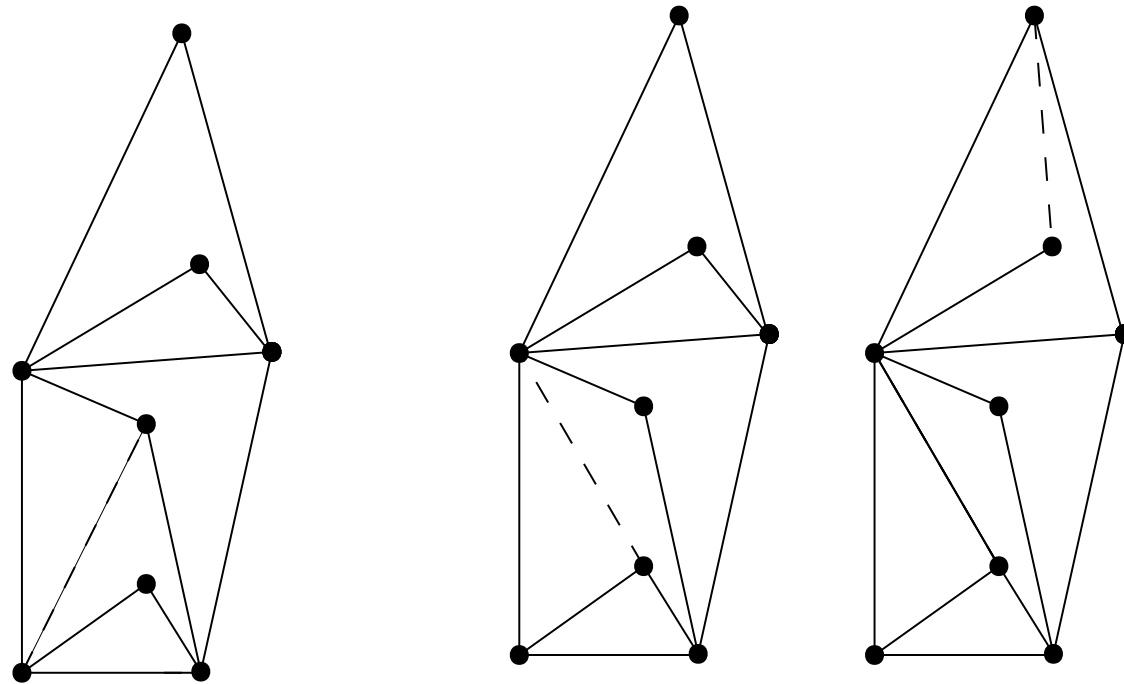
Lemma. *If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.*

Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.



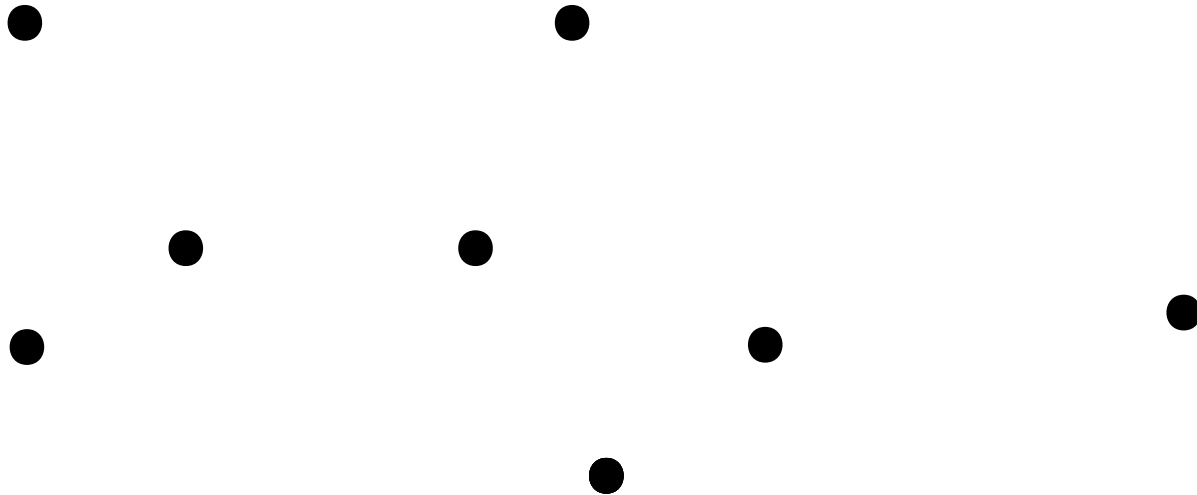
Can build a pseudotriangulation by adding shortest paths.

Can go from one pseudotriangulation to another by flipping edges.

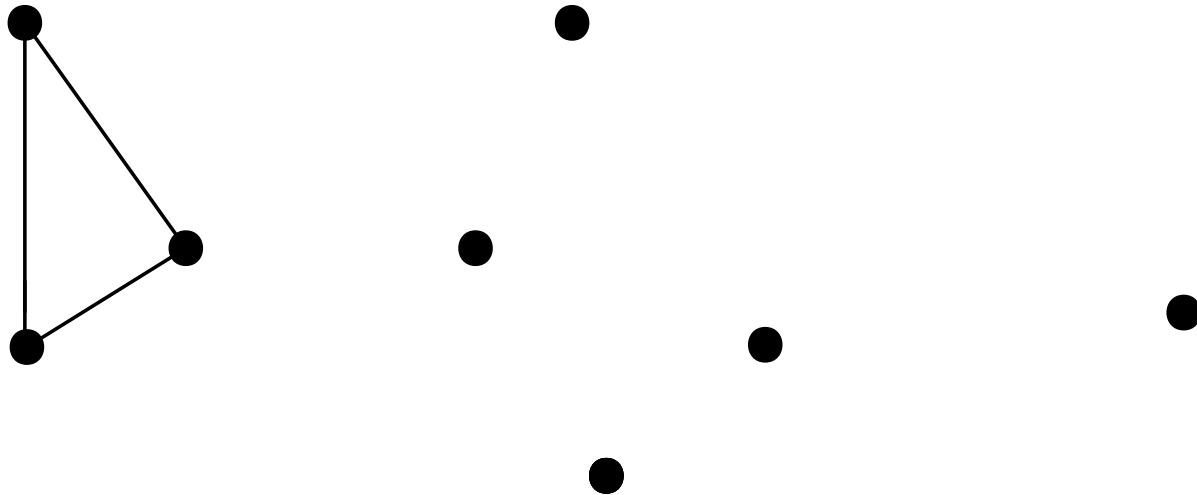


[Kettering, Snoeyink, Speckmann]

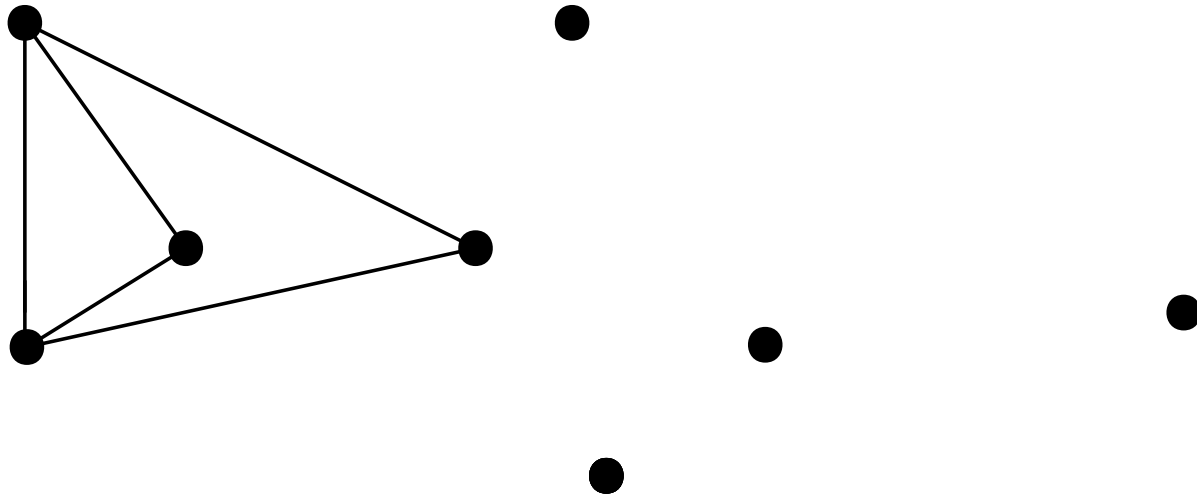
Another way to build a pseudotriangulation on a set of points is to keep adding vertices of degree two. Order by x-coord.



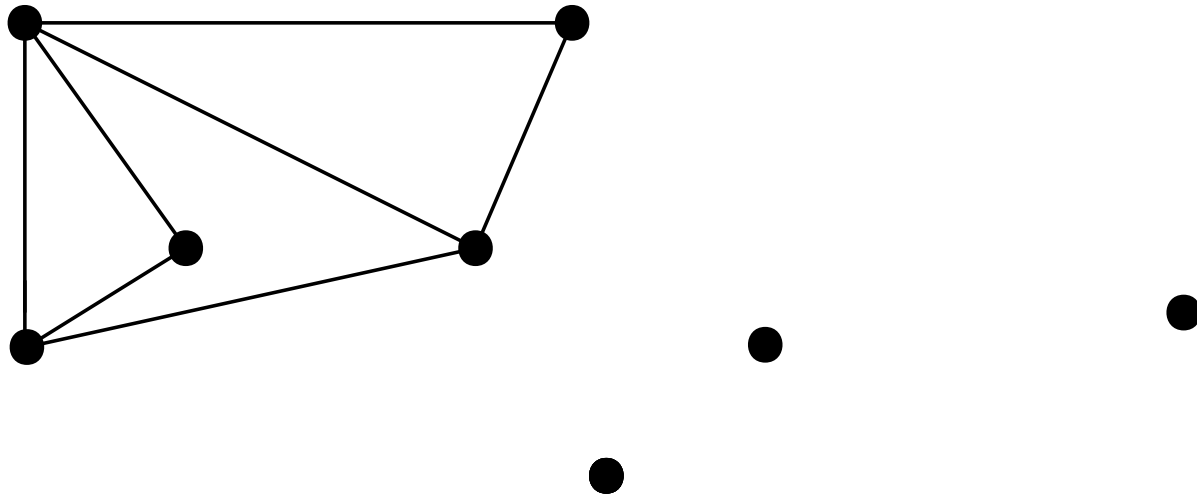
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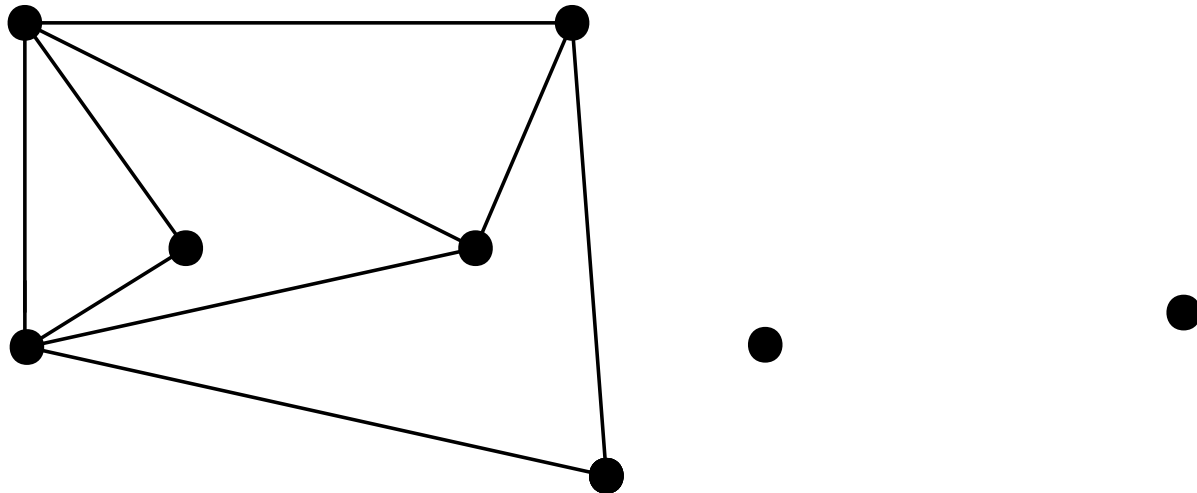
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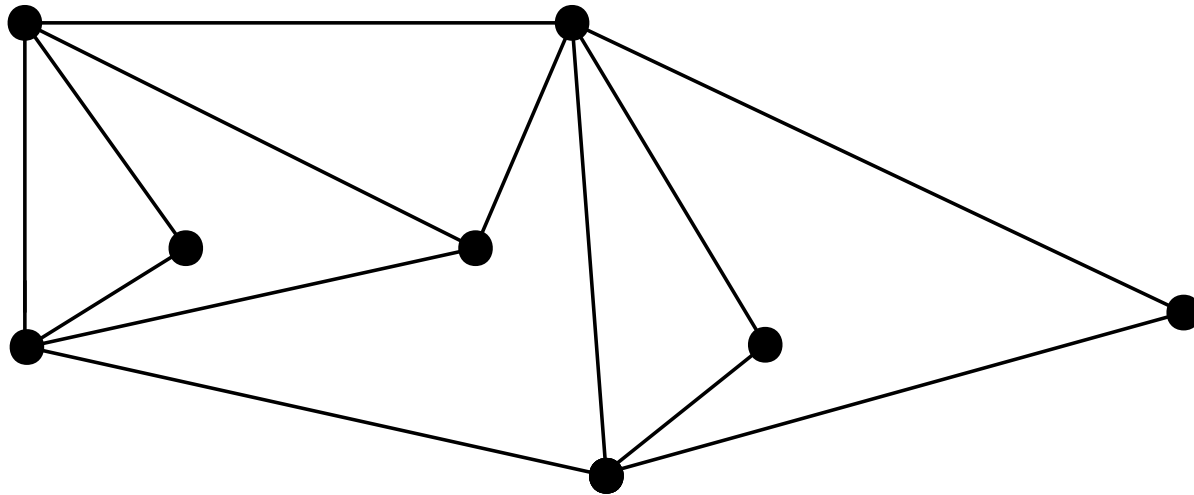
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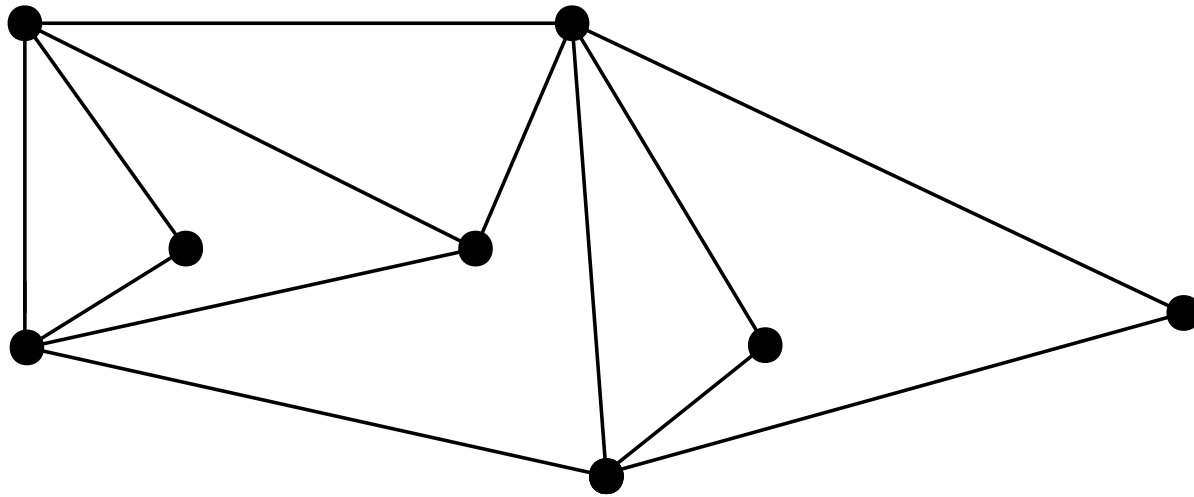
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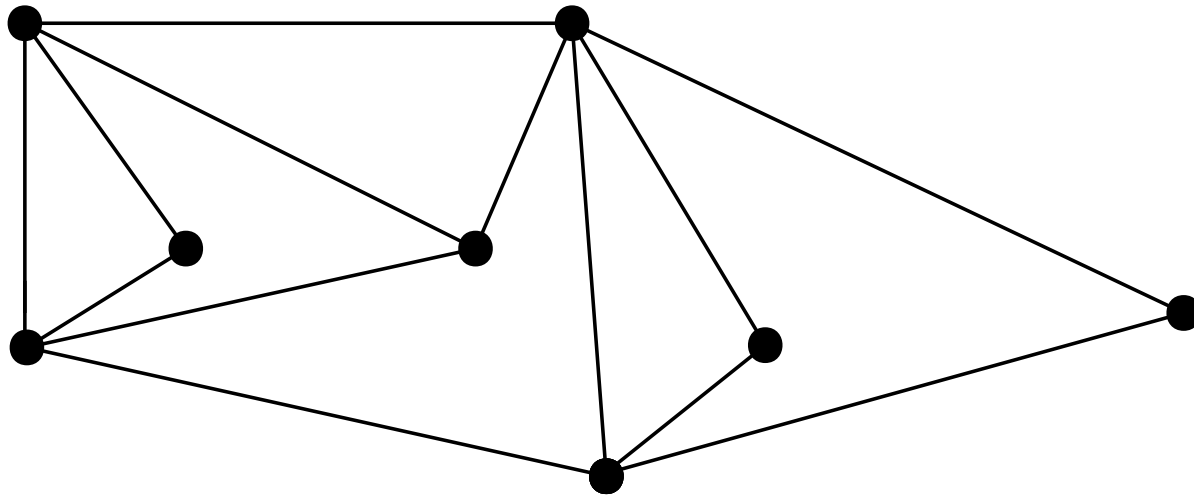
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So a minimal pseudotriangulation has $2(n-2)+1=2n-3$ edges.



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Indeed,

Theorem: Minimal pseudotriangulations are minimally rigid. [Streinu]

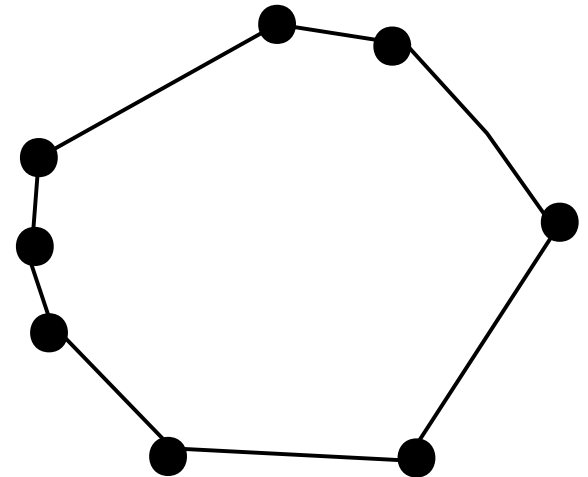
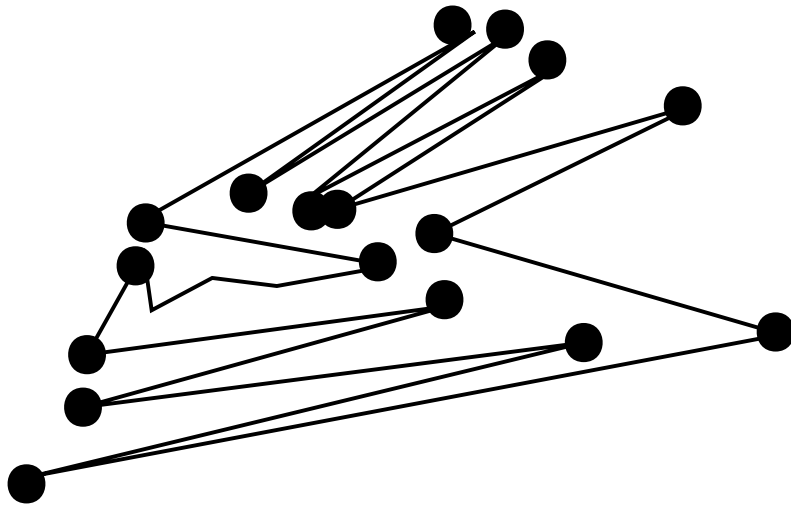
Part IV: Relationships between these.

Theorem: Minimal pseudotriangulations are minimally rigid.

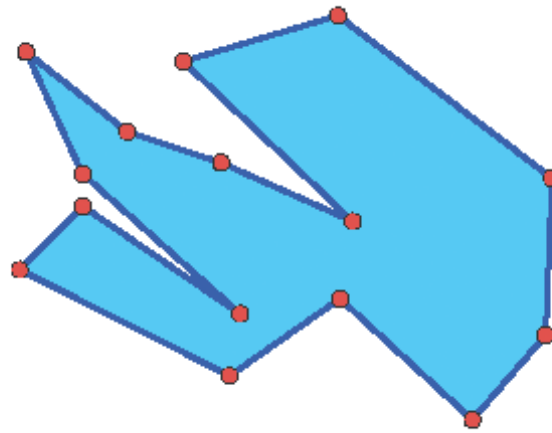
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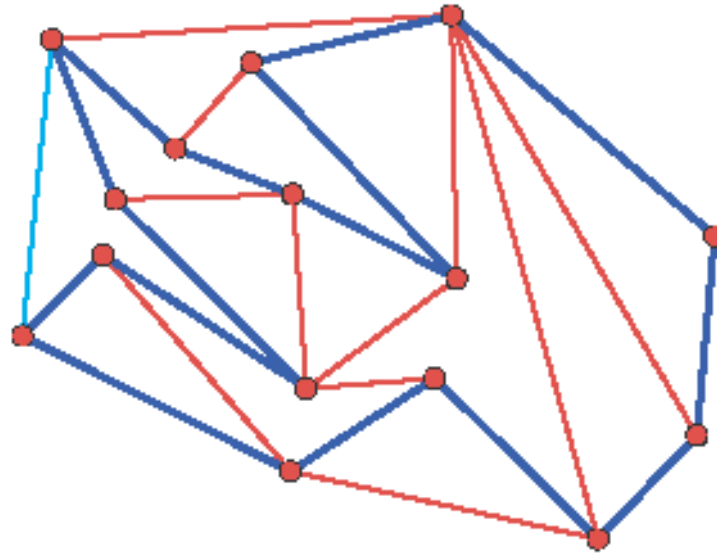
Pseudotriangulations provide structure to convexify embedded cycles.

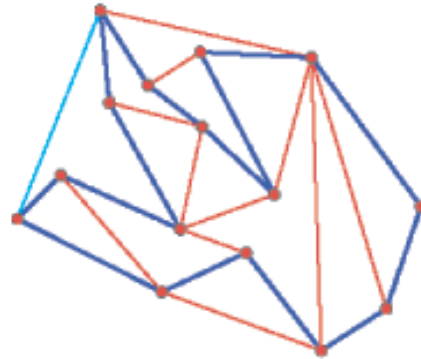
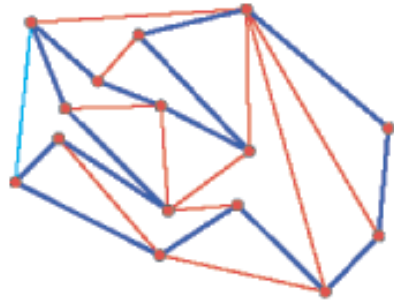


Algorithm for Convexifying:



Pseudotriangulate the polygon, including hull edges.

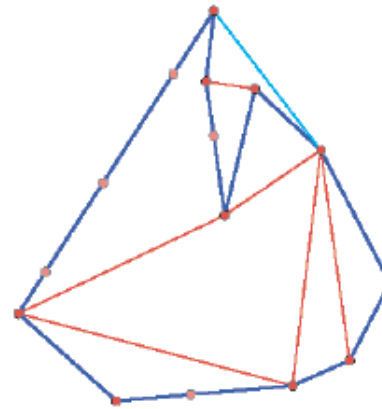
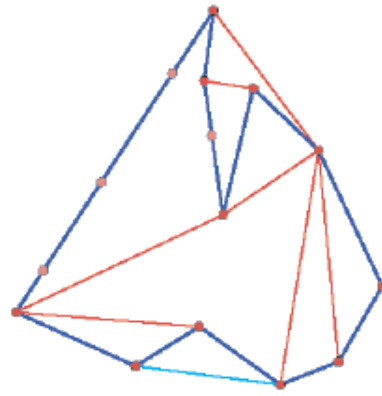




Remove one hull edge. Now have one-degree of freedom and its an *expansive motion*.

Open the object.

Repeat.



<http://cs.smith.edu/streinu/Research/Motion/motion.html>

<http://cs.smith.edu/streinu/Research/Motion/Animation/mech1a.html>

Part IV: Relationships between these.

So far:

- Theorem: Minimal pseudotriangulations are minimally rigid.
- Pseudotriangulations provide structure to convexify embedded cycles.

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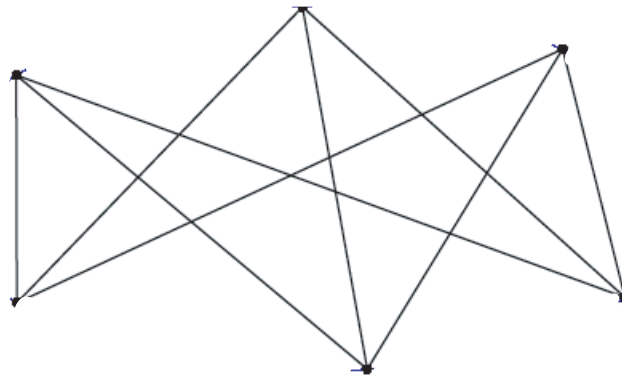
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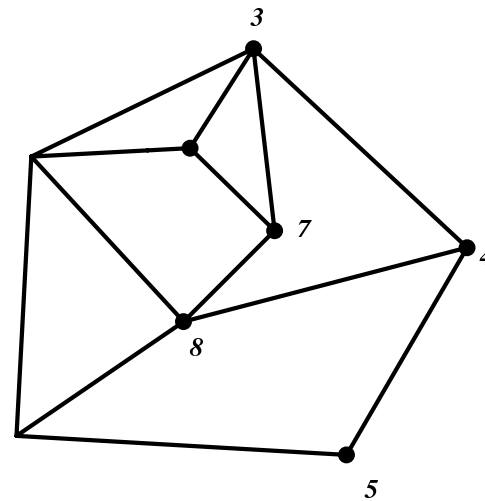
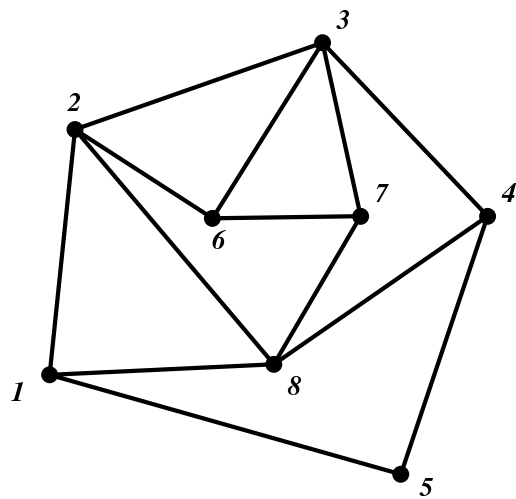
Next:

Are there rigid graphs that can not be embedded as pseudotriangulations?

Not every minimally rigid graph is planar:



Plane embeddings of minimally rigid graphs may not be pseudotriangulations



Pseudotriangulations are minimally rigid graphs. (Streinu)

Can every minimally rigid graph be embedded as a pseudotriangulation?

Pseudotriangulations are minimally rigid graphs. (Streinu)

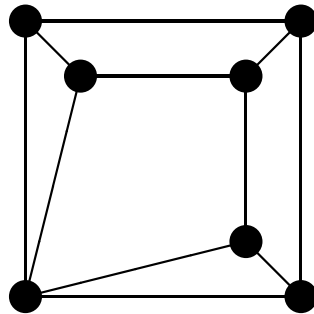
Can every minimally rigid graph be embedded as a pseudotriangulation?

Yes!

Theorem: Every planar minimally rigid graph can be embedded as a pseudotriangulation. (HORSSSSSW)

Characterizations of minimally rigid graphs.

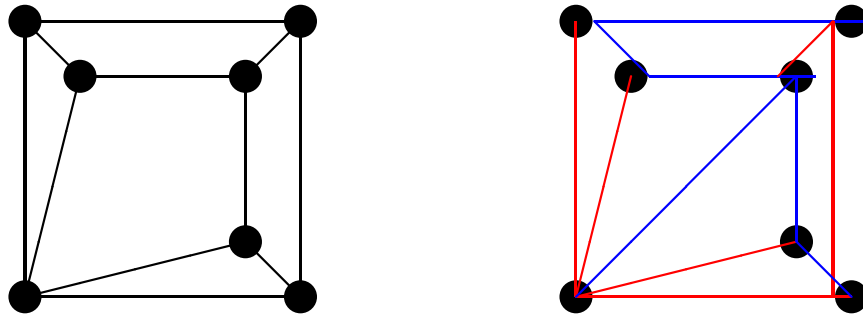
Theorem (Laman, 1970) A graph is minimally rigid if and only if $E_G = 2V_G - 3$ and for any subgraph H with at least 2 vertices, $E_H \leq 2V_H - 3$.



Algorithm? Not very efficient...

Characterizations of minimally rigid graphs.

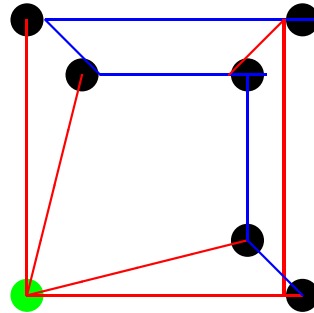
Theorem (Lovasz - Yemini, 1982) A graph is minimally rigid if and only if adding any edge (including doubling an existing edge) results in a graph that is two edge-disjoint spanning tree.



Algorithm? Edmonds Matroid Partitioning Algorithm
-polynomial in V, E . Run it $C(n,2)$ times!

Characterizations of minimally rigid graphs.

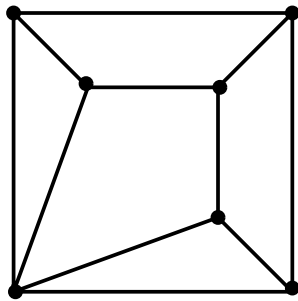
Theorem (Crapo, 1992) A graph is minimally rigid if and only if its edges are the disjoint union of 3 trees such that each vertex is incident with exactly 2 trees and in any subgraph the trees have different spans.



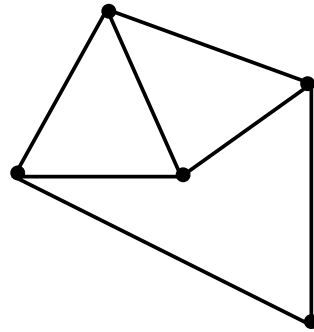
This characterization can be tested in one application of a modified Matroid partitioning algorithm.

Characterizations of minimally rigid graphs.

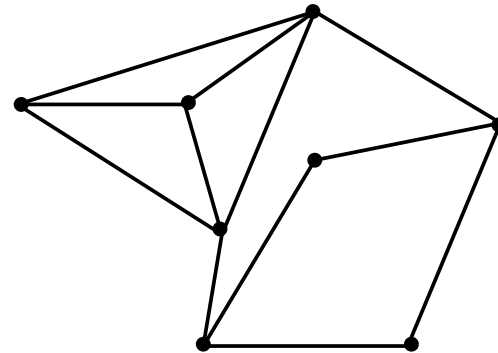
Theorem (Henneberg, 1860): A graph is minimally rigid if and only if it can be constructed using a **Henneberg 2-sequence**.



(i)

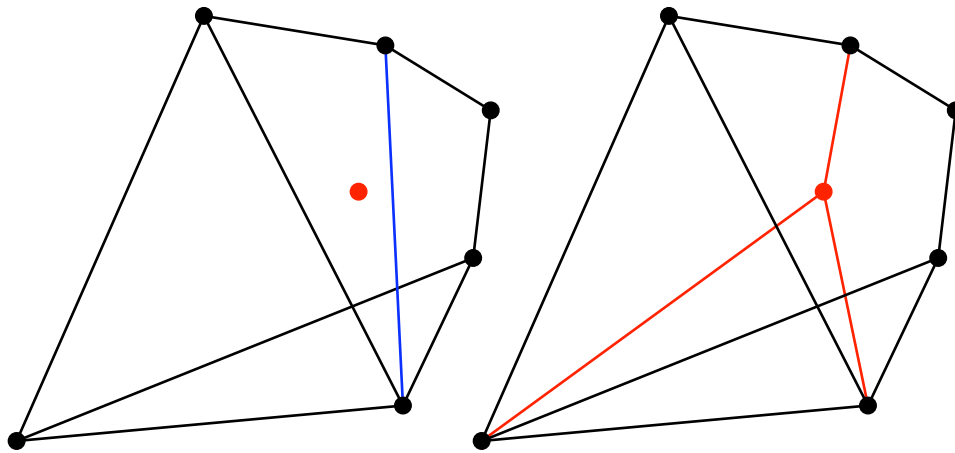


(ii)



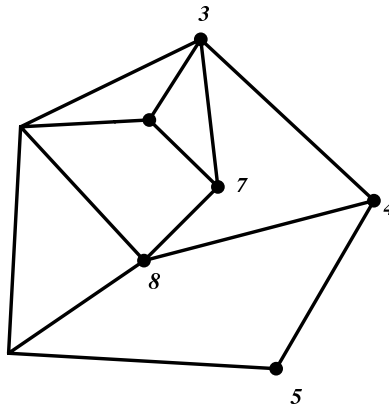
(iii)

2. Delete an edge $\{v_1, v_2\}$, and add a new vertex v adjacent to each of $\{v_1, v_2\}$ adjacent to any additional vertex.



Use Henneberg to prove any minimally rigid graph can be embedded as a pseudotriangulation.

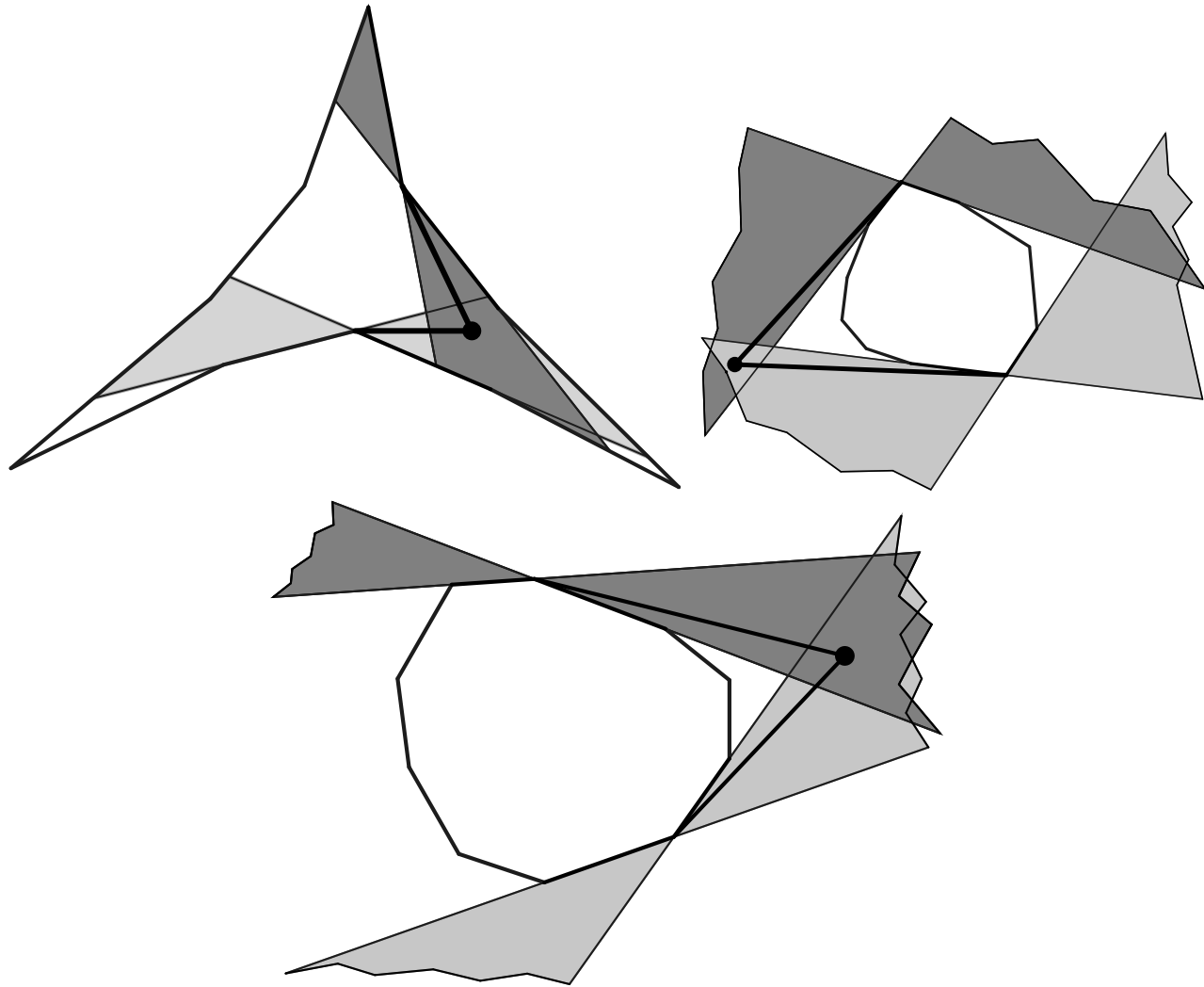
- Preprocess the vertices to find order they must be added using Henneberg moves.



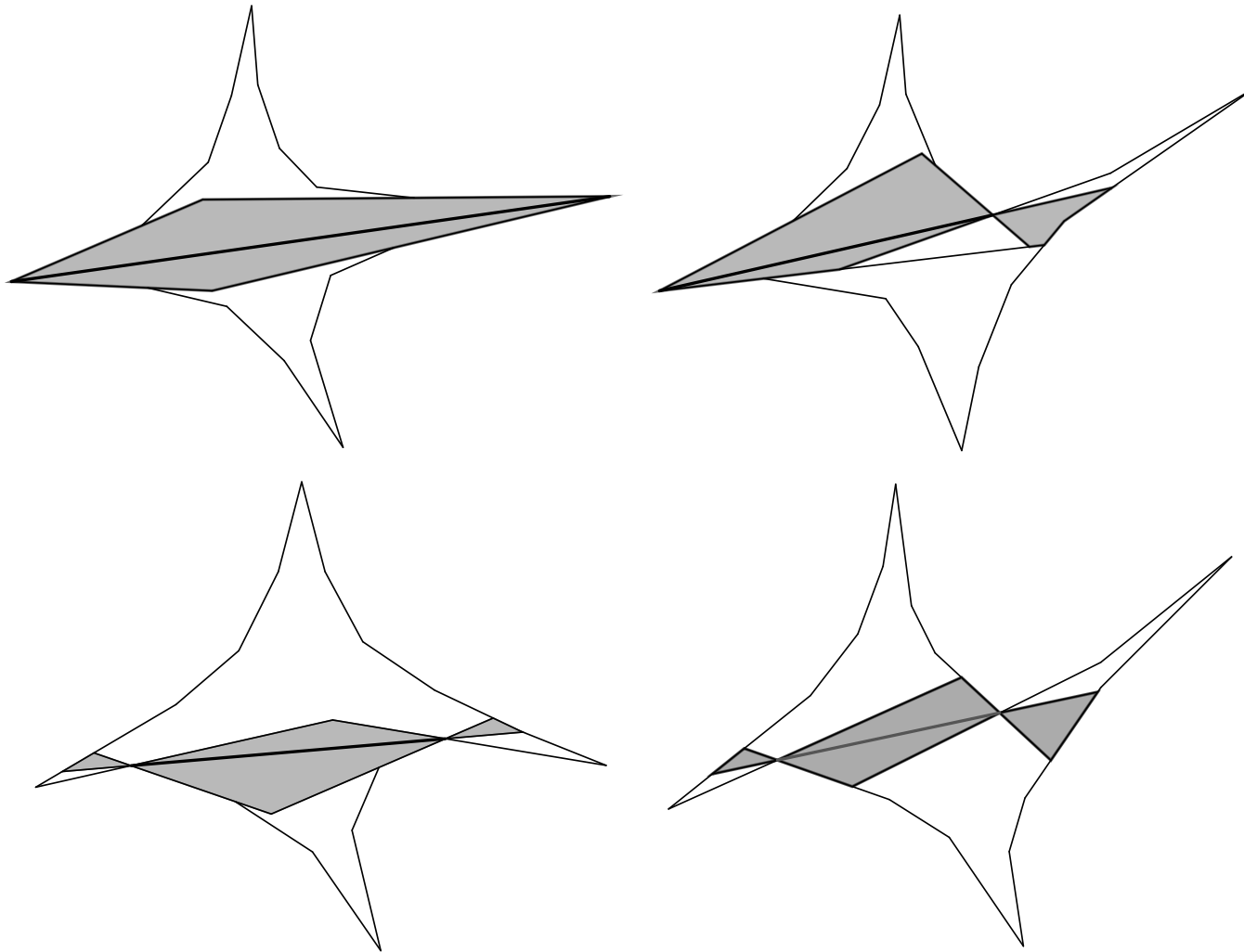
Vertex order: 2, 3, 6, 7, 8, 1, 4, 5

- Construct the PT. Maintain the pseudotriangulation at each step.

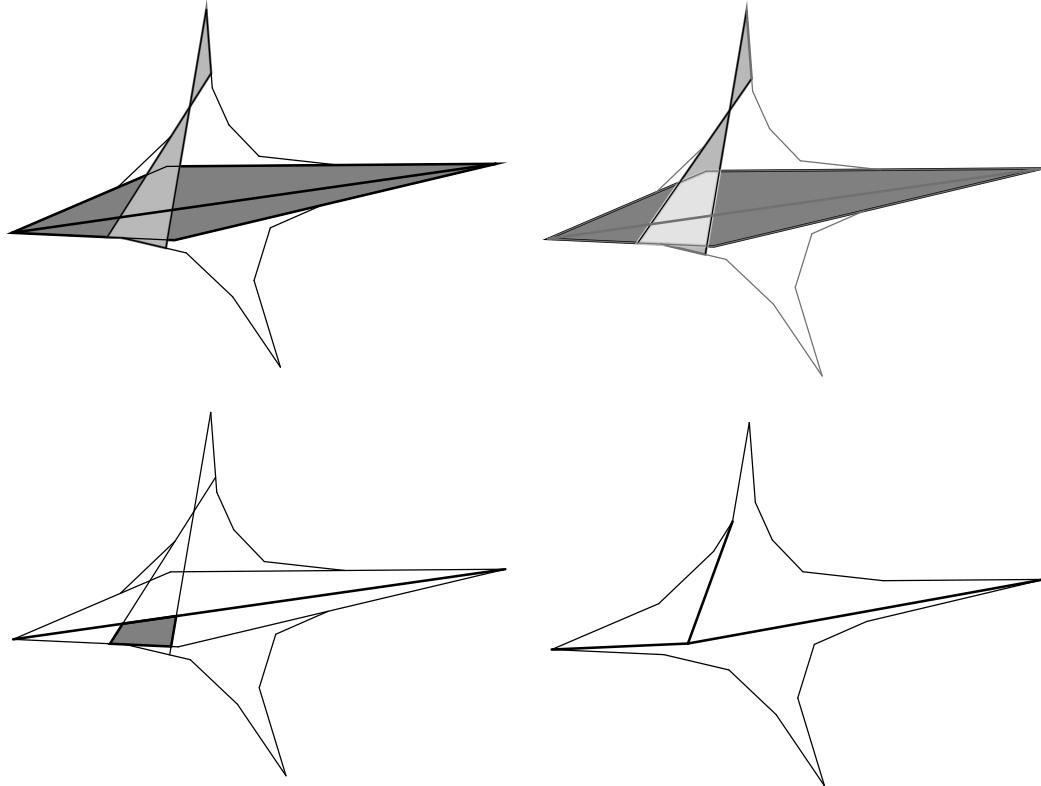
Adding a vertex of degree 2 to a pseudotriangle:



Henneberg type 2: Remove an edge: Get a pseudo-quad.



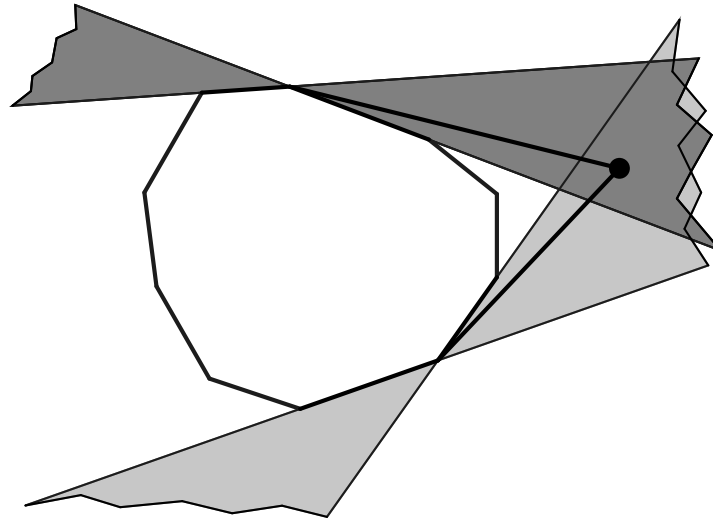
add a vertex of degree 3.



New vertex must also be pointed.

This always works.

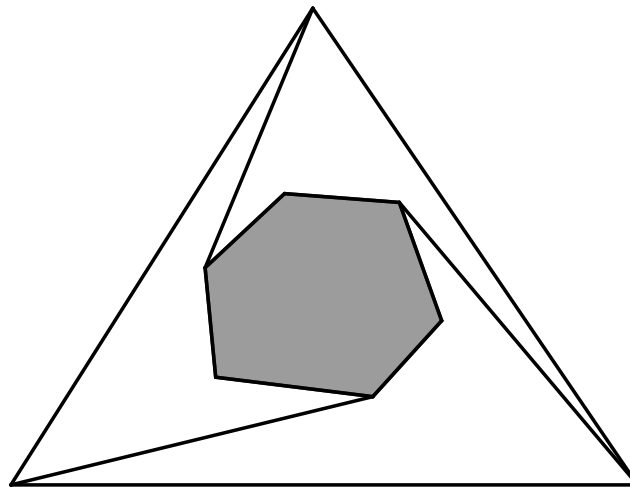
This does not let us maintain the combinatorial structure of the embedding.



It is algorithmic, of order $O(n^3)$.

Minor modifications allow us to:

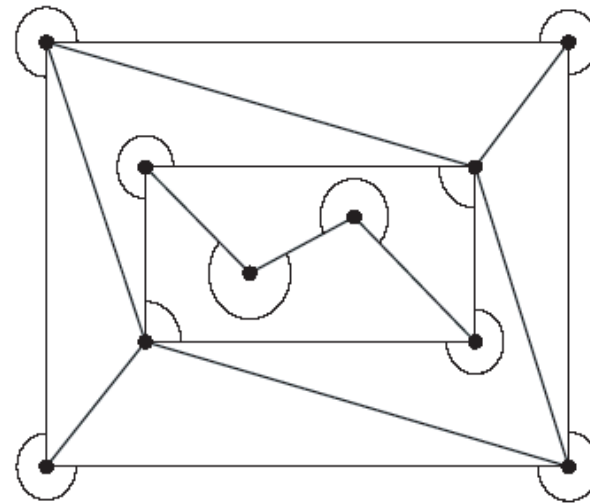
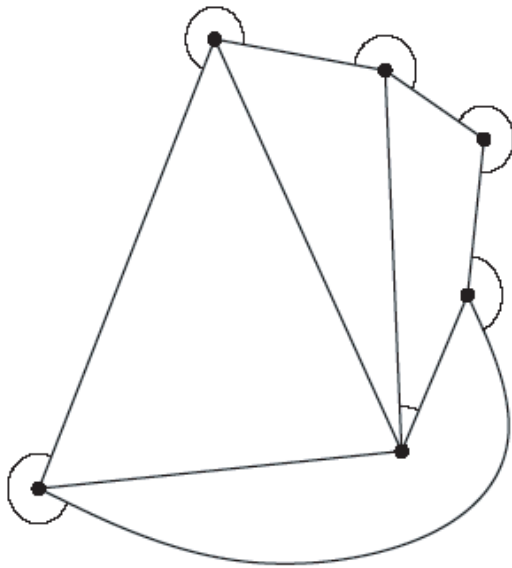
- Predetermine the faces.
- Predetermine the outside face.



A different proof technique lets us:

Realize any COMBINATORIAL pseudo-triangulation
(pre-assign which vertex is pointed in each face.)

Combinatorial pseudotriangulations



There are Other Applications of Pseudotriangulations in
Comp. Geom.

- Data structure for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink, '94] and visibility [Pocciola and Vegter, '96].
- Kinetic Collision detection [Agarwal, Basch, et. al. '01]; [Kirkpatrick, Snoeyink, Speckmann '00, '02]
- Art Gallery Problems [Pocciola and Vegter, '96] and [Speckmann and Toth, 01].

Current Work and Open Questions:

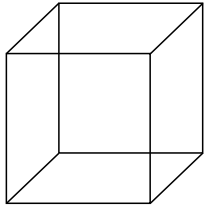
- Embedding Pseudotriangulations on a small grid.
 $O(n) \times O(n)$???
- Graph theoretic properties of pseudotriangulations.
- Pseudotriangulations in 3-space.

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- Embedding Pseudotriangulations on a small grid.
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- Pseudotriangulations in 3-space. (Rigidity is not well understood!)

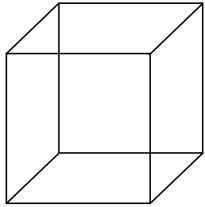
End.

Rigidity in 3-d.



Now every vertex has 3 degrees of freedom.

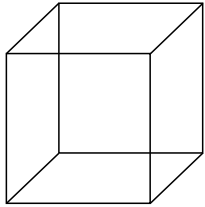
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There are 6 independent translations and rotations in \mathbb{R}^3

Rigidity in 3-d.

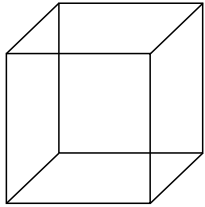


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Theorem: If a graph is **minimally rigid in \mathbb{R}^3** then $E_G = 3V_G - 6$ and for any subgraph H with at least 2 vertices, $E_H \leq 3V_H - 6$.

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Converse is false...

