

1.8 Exact solutions for differential equations: Separation of variables

Sometimes it is possible to find exact formulas for y given the formula for y' . We have already done this by just guessing in some cases. One better method to do this is called *separation of variables*. TO do this we take a formula for $\frac{dy}{dt}$ and “cross multiply” so that all t 's are on one side and all y 's on the other. Then take the integral of each side. We show the method by examples:

Example 1: $\frac{dy}{dt} = -32t$. Multiply both sides by dt to get $dy = -32tdt$. Then $\int dy = \int -32tdt$ so $y = -16t^2 + c$. We can use any initial conditions we have to find c . In the apple problem from chapter 1, at time 0, $y(0) = 100$. So $100 = -16(0)^2 + c$, and $c = 100$.

Example 2: If $\frac{dy}{dt} = ay$. Get all y 's to the same side: $\frac{1}{y}dy = adt$. Integrate both sides: $\int \frac{1}{y}dy = \int adt$ to get $\ln y = at + c$ where c is a constant that will depend on the initial conditions. We really want a formula for y in terms of t , so more work is needed here. Raise e to the power of each side:

$$e^{(\ln y)} = e^{(at+c)}$$

$$y = e^{(at+c)} = e^{at} e^c$$

Note that this is the answer we guessed at last time.

Example 3: $\frac{dy}{dx} = -\frac{x}{y}$. Separate to get $ydy = -xdx$. Integrate to get $\int ydy = -\int xdx$. And so, $\frac{y^2}{2} = -\frac{x^2}{2} + c$. So we get $y = \sqrt{2c - x^2}$. Note that a nicer way to look at this is that $x^2 + y^2 = 2c = C$.

Example 4: $\frac{dy}{dx} = \frac{x+y}{x-y}$. Separate to get $(x-y)dy = (x+y)dx$. This is not good enough! We must have only y 's on one side, and only x 's on the other. The technique of separation of variables is not good enough to solve every problem!

1.9 Problems about separation of variables:

In each of the following (i) use the initial value given to sketch $y(t)$ using Slinky. (ii) Use the technique of separation of variables to find an exact formula for $y(t)$.

Exercise 1.13. $\frac{dy}{dt} = .02y, y(0) = 300$.

Exercise 1.14. $\frac{dy}{dt} = .6y, y(0) = 50$.

Exercise 1.15. $\frac{dy}{dt} = \frac{5y}{t}, y(0) = 10$.

Exercise 1.16. $\frac{dy}{dt} = te^y, y(0) = 0$.

Exercise 1.17. $\frac{dy}{dt} = 3y + 8, y(0) = 25$.

Exercise 1.18. $\frac{dy}{dt} = y + y^2t, y(0) = 1$.

Exercise 1.19. $\frac{dy}{dt} = y(2 - y), y(0) = 1$.

Exercise 1.20. $\frac{dy}{dt} = \frac{y}{t(t+1)}, y(1) = 1$.