Chapter 5

Series and Convergence

We know a Taylor Series for a function is a polynomial approximations for that function. This week, we will see that within a given range of x values the Taylor series *converges* to the function itself. In order to fully understand what that means we must understand the notion of a limit, and convergence.

5.1 Limits of functions and L'hopital's rule

We begin with a brief review of limits. You have probably studied limits of functions before. Intuitively, $\lim_{x\to a} f(x) = L$ means the closer x gets to a the closer the value $f(x)$ gets to L. Indeed, as close as you want to get to L, say within .0001 you can find an x^* such that $|f(x^*) - L| \leq .0001$.

Some limits are easy to find. For example, $\lim_{x\to a} x = a$ and $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$. If we happen to know that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then it is true that:

(i)$\lim_{x\to a} f(x) + g(x) = L_1 + L_2$

(ii)$\lim_{x\to a} f(x)g(x) = L_1L_2$

(iii)$\lim_{x\to a} f(x)/g(x) = L_1/L_2$, if $L_2 \neq 0$.

You may recall that it is more difficult to find $\lim_{x\to a} f(x)/g(x)$ when $\lim_{x\to a} f(x) = 0, \infty$ or $-\infty$ and $\lim_{x\to a} g(x) = 0, \infty$ or $-\infty$. For example consider $\lim_{x\to\infty} \dfrac{5x^5 - 3x}{7x^5 - 13x}$. Both the numerator and denomonator go to ∞, but the fraction goes to 5/7. One way to see this is to graph the function! Here's another method.

Theorem (L'Hopital's rule): If $\lim_{x\to a} f(x)/g(x)$ is of indeterminate form ($0/0$ or $\pm\infty/\pm\infty$) then

$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x)$$

Here are some examples.

$$\lim_{x\to\infty} (3x^2 + 4)/(7x - 23) = \lim_{x\to\infty} 6x/7 = \infty.$$

$$\lim_{x\to 2} (3x^2 + 4x - 20)/(x^3 - 8) = \lim_{x\to 2} (6x + 4)/(3x^2) = 4/3.$$

We can now make a general statement about limits of quotients of polynomials.

Theorem: If $f(x)$ and $g(x)$ are polynomials then:

$$\lim_{x\to\infty} f(x)/g(x) = \begin{cases} \infty & \text{if deg } f(x) > g(x) \\ 0 & \text{if deg } f(x) < g(x) \\ a_k/b_k & \text{if deg } f(x) = g(x) \end{cases}$$

where a_k and b_k are the coefficients of the highest terms in $f(x)$ adn $g(x)$ respectively.

5.2 Limits of sequences

A *sequence* is a function whose domain is all positive integers. For example, $2, 4, 8, 16, \ldots$ is a sequence whose nth term is 2^n. We write this as the sequence $a_n = 2^n$. List the first few terms of the sequence $\frac{1}{3n}$, what is the 34th term of this sequence?

In this class we will mostly be interested in limits of sequences as $x \to \infty$. It is not too hard to believe that $\lim_{n\to\infty} 2^n = \infty$, and that $\lim_{n\to\infty} \dfrac{1}{3n} = 0$. Intuitively, we understand that as n gets really big, then $\frac{1}{3n}$ gets really close to 0. Now we try to formalize that idea.

Definition: $\lim_{n\to\infty} a_n = L$ if for any $\epsilon > 0$ there exists an integer N so that whenever $n > N$ it is true that $|a_n - L| < \epsilon$. We say the sequence a_n *converges* to L in this case.

For example, $\lim_{n\to\infty} \dfrac{1}{3n} = 0$ since, whenever $n > 1/(3\epsilon)$. I.e., inorder to make $\frac{1}{3n}$ within $\epsilon = .001$ of 0 we need to pick $n > \frac{1}{3(.001)} = 333$.

5.3 Series

A series is an infinite sum, like

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots$$

Taylor Series are examples of series. In this section we address the following question. If we look at partial sums of the series, do the answers we get approach some limit. I.e.,

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i = ?$$

In other words does the sequence of partial sums $a_n = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i$ converge to some limit? We say that a series *converges* if such a limit exists and is finite and *diverges* otherwise. Here are three examples, the first series converges and the second and third diverge.

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

$$\sum_{i=0}^{\infty} i = 1 + 2 + 3 + 4 + 5 + \cdots$$

$$\sum_{i=0}^{\infty} (-1)^i = -1 + 1 - 1 + 1 - 1 + 1 \cdots$$

In order for a series to converge it must be true that the terms in it get smaller and smaller. More exactly:

Theorem: If $\sum_{i=0}^{\infty} b_i$ converges, then $\lim_{i\to\infty} b_i = 0$.

This condition is needed for a series to converge but is not sufficient to insure convergence! So if we have a series $\sum_{i=0}^{\infty} b_i$, and $\lim_{i\to\infty} b_i \neq 0$ then the series must diverge. The second and third examples above are examples of this. But it is possible that a series $\sum_{i=0}^{\infty} a_i$, has $\lim_{i\to\infty} a_i = 0$ and doesn't converge! A surprising example is:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

5.4 The Geometric Series

The first series we will talk about is called the geometric series. It is of the form

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + x^4 \cdots$$

Notice that the very first series mentioned at the top of this page is such a series with $x = \frac{1}{2}$. Whether this series converges or not will depend on what x is.

We first look at the simple case that $x = \frac{1}{2}$. It is useful to define the partial sums here and study there behavior.

$$S_n = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i = ?$$

Calculating these we see, $S_0 = 1$, $S_1 = \frac{3}{2}$, $S_2 = \frac{7}{4}$, $S_3 = \frac{15}{8}$, $S_4 = \frac{31}{16}, \cdots$. We see a pattern! Indeed

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{2^{n+1}-1}{2^n}\right) = 2 - \lim_{n\to\infty} \left(\frac{1}{2^n}\right) = 2$$

For the general geometric series, we again look at partial sums.

$$S_n = \sum_{i=0}^{n} x^i = 1 + x + x^2 + \cdots + x^n$$

In this case, some algebra proves useful. Consider multiplying the partial sum by $(1 - x)$.

$$(1-x)S_n = (1-x)(1 + x + x^2 + \cdots x^n) = 1 - x^{n+1}$$

This means that

$$S_n = \frac{1 - x^{n+1}}{1 - x}$$

So we need to determine the limit

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x}$$

Notice that it is a limit as n goes to infinity. The only n is in the numerator. Whether this converges or not will depend on what x is. In the example when $x = \frac{1}{2}$ the series converges. When $x = 2$ on the otherhand, the series will diverge. The general rule is that the series will converge as long as $|x| < 1$. Why?

So if $|x| < 1$

$$\sum_{i=0}^{\infty} x^i = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}$$

Check that this corresponds with what we got when $x = \frac{1}{2}$.

Repeating decimals   You all know that $.3333333333\ldots = \frac{1}{3}$, but here is a proof of this fact.

$$.3333333 = .3 + .03 + .003 + .0003 + \cdots = 3 * \frac{1}{10} + 3 * \frac{1}{100} + 3 * \frac{1}{1000} + \cdots = 3 \sum_{i=1}^{\infty} \left(\frac{1}{10}\right)^i$$

$$= 3\left(\frac{1}{1-.1} - 1\right) = 3\left(\frac{1}{9}\right) = \frac{1}{3}$$

5.5 Alternating Series

An alternating series is one in which the terms alternate in sign, so it will look like $\sum_{n=1}^{\infty}(-1)^n b_n$ where b_n will be sequence. The following theorem about alternating series will be useful.

Theorem: An alternating series $\sum_{i=0}^{\infty}(-1)^i b_i$ converges if and only if $\lim_{i\to\infty} b_i = 0$.

For example, the series $\sum_{i=0}^{\infty}(-1)^i \frac{1}{i}$ converges, while the series $\sum_{i=0}^{\infty}(-1)^i 2^{\frac{1}{n}}$ diverges. Compare this result with the previous theorem. This one is "if and only if", while the previous theorem was not.

5.6 Tests for Convergence and Absolute Convergence

We continue with more ways of determining whether a series converges or not. Since we already have a method which determines whether alternating series converge or diverge, this week we will concentrate on series of positive terms.

5.6.1 Method 1: Comparison

(I) If $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are series of positive terms and $a_i \leq b_i$ and the series $\sum_{i=0}^{\infty} b_i$ converges, then so does the series $\sum_{i=0}^{\infty} a_i$.

(II) If $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are series of positive terms and $a_i \geq b_i$ and the series $\sum_{i=0}^{\infty} b_i$ diverges, then so does the series $\sum_{i=0}^{\infty} a_i$.

Example: Show that the series $\sum_{i=0}^{\infty} \frac{1}{(i+1)!}$ converges. We compare this series with the series $\sum_{i=0}^{\infty} \frac{1}{2^i}$, which is a geometric series that converges to 2. We compare the ith terms:

$$\frac{1}{(i+1)!} = \frac{1}{1 \cdot 2 \cdot 3 \cdots i \cdot (i+1)} \leq \frac{1}{(1 \cdot 2 \cdot 2 \cdot 2 \cdots 2)} = \frac{1}{2^i}$$

5.6.2 Method 2: Integral Test

If each $a_i = f(i)$ for some continuous decreasing function $f(x)$ then

$$\sum_{i=1}^{\infty} a_i \text{ converges if and only if } \int_1^{\infty} f(x)dx \text{ converges.}$$

Example: Show that the series $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges. The function $f(x) = \frac{1}{x}$ is continuous and decreasing on the interval $(1, \infty)$ and $a_i = \frac{1}{i}$.

$$\int_1^{\infty} \frac{1}{x}dx = \lim_{m\to\infty} \int_1^{m} \frac{1}{x}dx = \lim_{m\to\infty} (\ln|m| - \ln 1) = \infty$$

Example: The series $\sum_{i=1}^{\infty} \frac{1}{i^p}$ converges. The function $f(x) = \frac{1}{x^p}$ is continuous and decreasing on the interval $(1, \infty)$ and $a_i = \frac{1}{i^p}$.

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{m\to\infty} \int_1^m \frac{1}{x^p} dx = \lim_{m\to\infty} \left(\frac{1}{1-p}\right)\left(\frac{1}{m^{1-p}} - 1\right)$$

This limit is finite if $p > 1$ and infinite otherwise.

Theorem: The series $\sum_{i=1}^{\infty} \frac{1}{i^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Absolute Convergence Sometimes a series is neither all positive term nor nicely alternating in sign. An example is $\sum_{i=1}^{\infty} \frac{\cos i}{i^2}$, some of whose terms are negative, some positive but not every other one. In this case we sometimes talk about the *absolute convergence* of a series.

The series $\sum_{i=1}^{\infty} a_i$ is said to converge absolutely if the series of the absolute values of the terms

$$\sum_{i=1}^{\infty} |a_i| = |a_1| + |a_2| + \cdots$$

converges. It is important because of the following result:

Theorem: If a series converges absolutely then it converges.

Example: Looking at our example $\sum_{i=1}^{\infty} \frac{\cos i}{i^2}$ must converge since $\sum_{i=1}^{\infty} \left|\frac{\cos i}{i^2}\right|$ converges (by comparison with $\sum_{i=1}^{\infty} \frac{1}{i^2}$).

5.6.3 Method 3: Ratio Test

This test is a generalization of the comparison test above. This tests for absolute convergence.

Theorem:

$$\sum_{i=1}^{\infty} a_i \text{ converges if and only if } \lim_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} < 1.$$

Example: Recall the Geometric series is $1 + x + x^2 + \cdots + c^i + \cdots$. The ratio test looks at the ratio of the terms $a_{i+1} = x^{i+1}$ and $a_i = x^i$:

$$\lim_{i\to\infty} \frac{|a_{i+1}|}{|a_i|} = \lim_{i\to\infty} \frac{|x^{i+1}|}{|x^i|} = |x|.$$

This limit exists and is < 1 exactly when $|x| < 1$. Thus the geometric series converges when $|x| < 1$ which agrees with what we had determined before.

5.7 Intervals of Convergence

This brings us to another definition. Series that contain a variable x, say, may converge for only some values of x, The values for which the series does converge are collectively called the *interval of convergence* for that series. For example, the geometric series has interval of convergence $-1 < x < 1$. Returning to the Taylor series, we often want to know for what values of x does the Taylor Series of the function converge to the function itself. This will always turn out to be an interval around the point a where we centered the Taylor Series.

Examples:

(i) $\sum_{n=1}^{\infty} n^n x^n$. The ratio test give us that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(n+1)^(n+1)x^{n+1}|}{|n^n x^n|} = \lim_{n\to\infty} \frac{|(n+1)^(n+1)x|}{|n^n|} > \lim_{n\to\infty} \frac{|(n)^(n+1)x|}{|n^n|} > \lim_{n\to\infty} |nx|$$

which is ∞ unless $x = 0$. So the series converges only if $x = 0$.

(ii) $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{2n!}$ The ratio test shows:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(-2)^{n+1}x^{n+1}}{2(n+1)!}}{\frac{(-2)^n x^n}{2n!}} = \lim_{n\to\infty} \frac{-2x}{2(n+1)} = 0 < 1.$$

So this series converges for all x. That is, the interval of convergence is $-\infty < x < \infty$.

5.8 Problems for Chapter 5

Exercise 5.1. Find the limit in each case.

(a) $\lim_{x\to\infty} \frac{2x^3 - 3x^2}{7x - 13}$ (b) $\lim_{x\to\infty} \frac{4x^2 + 3x}{16x^3 - 1000}$ (c) $\lim_{x\to\infty} \frac{3x + \sqrt{x}}{2x}$

(d) $\lim_{x\to\infty} \frac{\ln(x)}{x}$ (e) $\lim_{x\to\infty} \frac{2^x}{x^{3567892}}$ (f) $\lim_{x\to 0} \frac{\sin(x)}{x}$

Exercise 5.2. Each of the following sequences converge to 0. In each case find N (as a function of ϵ) as the formal definition of convergence requires.

(a) $a_n = \frac{1}{n^2}$ (b) $a_n = \frac{1}{4n^3}$ (c) $a_n = \frac{1}{2^n}$ (hint: use logs)

Exercise 5.3. Decide if each of these geometric series converges and if so determine what it converges to.

(a) $100 + 1000 + 10000 + \cdots$ (b) $\sum_{i=0}^{\infty} \left(\frac{5}{9}\right)^i$ (c) $\sum_{i=0}^{\infty} \left(\frac{9}{5}\right)^i$

Exercise 5.4. . Find the rational numbers represented by each of the repeating decimals below.

(a) .040404040404 (b) .123123123123123...

Exercise 5.5. Find a formula for the sum of each of the following series by performing suitable operations on the geometric series.

(a) $1 - x^3 + x^6 - x^9 + \cdots$

(b) $x^2 + x^6 + x^{10} + \cdots$

(c) $1 - 2x + 3x^2 - 4x^3 + \cdots$

(d) $x + 2x^2 + 3x^3 + 4x^4 + \cdots$

(e) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

Exercise 5.6. Which of the following alternating sums converge and which diverge?

(a) $\sum_{i=1}^{\infty} (-1)^i \frac{i}{i^2+1}$ (b) $\sum_{i=1}^{\infty} (-1)^i \frac{1}{\sqrt{3i+2}}$ (c) $\sum_{i=1}^{\infty} (-1)^i \frac{i}{\ln i}$

(d) $\sum_{i=1}^{\infty} (-1)^i \frac{i}{5i-4}$ (e) $\sum_{i=1}^{\infty} (-1)^i \frac{1}{\ln i}$

Exercise 5.7. Use the ratio test to find the interval of convergence for each of the following power series.

(a) $\sum_{i=1}^{\infty} \frac{x^i}{i}$ (b) $\sum_{i=1}^{\infty} \frac{x^i}{2^i}$ (c) $\sum_{i=1}^{\infty} \frac{x^i}{i!}$

(d) $\sum_{i=1}^{\infty} i x^i$ (e) $\sum_{i=1}^{\infty} \frac{i^2}{2^i} x^i$ (f) $\sum_{i=1}^{\infty} i^{3i}$

(g) $\sum_{i=1}^{\infty} (1+x)^i$ (h) $\sum_{n=1}^{\infty} \frac{99}{n^n} x^n$ (i) $\sum_{n=1}^{\infty} n! x^n$

Exercise 5.8. Find the interval of convergence for the Taylor series of the following functions.

(a) $\sin(x)$ (b) e^x (c) $\cos(x)$

Exercise 5.9. Use the Comparison Test to determine whether each of these series converges or diverges.

(a)$\sum_{i=1}^{\infty} \frac{1}{i^2+i-1}$ (b)$\sum_{i=2}^{\infty} \frac{1}{\ln i}$ (c) $\sum_{i=0}^{\infty} \left(\frac{9}{5}\right)^i$

Exercise 5.10. Use the integral test to determine if each of the following series converges or diverges.

(a) $\sum_{i=1}^{\infty} \frac{i}{(i^2+1)}$ (b) $\sum_{i=1}^{\infty} \frac{1}{i^{.3}}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ (e) $\sum_{i=1}^{\infty} \frac{1}{i \ln(i)}$ (f) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Exercise 5.11. Use the ratio test to find the interval of convergence for each of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (b) $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ (c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

(d) $\sum_{n=1}^{\infty} n x^n$ (e) $\sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n$ (f) $\sum_{n=1}^{\infty} \frac{n^3}{x^n}$

(g) $\sum_{n=1}^{\infty} (1+x)^n$ (h) $\sum_{n=1}^{\infty} \frac{9^n}{n^9} x^n$ (i) $\sum_{n=1}^{\infty} n! x^n$

Exercise 5.12. Find the interval of convergence for the Taylor series of the following functions.

(a) $\sin(x)$ (b) e^x (c) $\cos(x)$