Math 225 class notes

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Fubini's Theorem

Suppose $A \subset \mathbb{R}^k$ and $f : A \to \mathbb{R}$ is integrable. Then $\int_A f = \sup\{L(f, P)\} = \inf\{U(f, P)\}$. Let $\underline{\int}_A = \sup\{L(f, P)\}$ and $\overline{\int}_A = \inf\{U(f, P)\}$. So in this new notation, we know we always have $\underline{\int}_A f \leq \overline{\int}_A f$ and f is integrable over A iff $\overline{\int}_A f = \underline{\int}_A f$.

Theorem 1 (Fubini's Theorem)**.** *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be closed rectangles, and $f : A \times B \to \mathbb{R}$ be integrable[1] Given an $x \in A$, let $g_x : B \to \mathbb{R}$ be defined by $g_x(y) = f(x, y)$. Let*

$$\mathcal{L}(x) = \underline{\int}_B g_x = \underline{\int}_B f(x, y)dy,$$

$$\mathcal{U}(x) = \overline{\int}_B g_x = \overline{\int}_B f(x, y)dy.$$

Then $\mathcal{L}$ and $\mathcal{U}$ are integrable on A and

$$\int_{A\times B} f = \int_A \mathcal{L} = \int_A \left(\underline{\int}_B f(x, y)dy \right) dx$$

$$\int_{A\times B} f = \int_A \mathcal{U} = \int_A \left(\overline{\int}_B f(x, y)dy \right) dx$$

Remark 1.

1. Remember that $\mathcal{L}(x) = \sup\{L(g_x, P)\})$ and $\mathcal{L}(x) = \inf\{U(g_x, P)\})$.
2. The point of the theorem is that the integral over $A \times B$ may be thought of in terms of "slices". The fact that the order doesn't matter is an easy consequence (see next).
3. A similar proof shows

$$\int_{A\times B} f = \int_B \mathcal{L} = \int_B \left(\underline{\int}_A f(x, y)dy \right) dx = \int_B \mathcal{U} = \int_B \left(\overline{\int}_A f(x, y)dy \right) dx$$

[1] I don't think we actually need to assume this to use the theorem, sometimes we can fudge things to work, see the remarks.

4. If g_x is not integrable at a finite number of points x, then we simply define $\mathcal{L}(x)$ to be 0 there. In this case $\int_A \mathcal{L}$ is unchanged. (This mostly works, but there are some exceptions, see Spivak p. 60.)

5. If $C \subset A \times B$, then $\int_C f := \int_{A\times B} f\chi_C$.

6. We can keep repeating the process on each of the nested integrals and so get the usual iterated integral from Multivariable Calculus. Namely if $A = [a_1, b_1] \times [a_2 \times b_2] \times \cdots \times [a_n, b_n]$ and $f : A \to \mathbb{R}$ is integrable, then

$$\int_A f = \int_{a_n}^{b_n} \left(\cdots \left(\int_{a_1}^{b_1} f \, dx_1 \right) \cdots \right) dx_n$$

Proof. □