Math 225 class notes

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Differentiation part 2

1 Differentiability and partial derivatives

Definition 1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x = a$ if there is a linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||} = 0.$$

The linear map λ is often denoted $f'(a)$ and the matrix which represents the linear map is often denoted $Df(a)$.

So $f'(a) : \mathbb{R}^n \to \mathbb{R}^m$ is $f'(a)(h) = f'(a)(h_1, h_2, \ldots, h_n)$, or $Df(a)$ is an $m \times n$ matrix.

Definition 2. If $f : \mathbb{R}^n \to \mathbb{R}$and $a \in \mathbb{R}^n$ then the i**th partial derivative** of f at a is

$$\lim_{h \to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n - f(a_1, \ldots, a_n)}{h}$$

provided this limit exists. It is denoted $D_i f(a)$ or $\frac{\partial f}{\partial x_i}(a)$.

Theorem 1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f^i(a) = \frac{\partial f^i}{\partial x_j}(a)$ exists for $1 \leq i \leq m$ and $1 \leq j \leq n$ and $f'(a) = Df(a)$ is the $m \times n$ matrix whose ijth entry is $D_j f^i(a) = \frac{\partial f^i}{\partial x_j}(a)$.*

Definition 3. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then the **Jacobian matrix** $Df(a)$ is the $m \times n$ matrix whose ijth entry is $D_j f^i(a) = \frac{\partial f^i}{\partial x_j}(a)$.

Remark: This theorem says that if f is differentiable at a then the partial derivatives **exist** at a, it does **not** say the derivatives are continuous at a. Of course the partial derivatives may be continuous at a, but they don't have to be. Examples of such functions are found in HW 6 question 1. Namely,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at 0, but $f'(x)$ is not continuous at 0.

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$, but neither $\frac{\partial f}{\partial x}$ nor $\frac{\partial f}{\partial y}$ is continuous at $(0,0)$.

Theorem 2. *If $f : \mathbb{R}^n \to \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f^i(x) = \frac{\partial f^i}{\partial x_j}(x)$ exist in an open set containing a and if each function $D_j f^i = \frac{\partial f^i}{\partial x_j}$ is continuous at a.*

Definition 4. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called **continuously differentiable at** a if $D_j f^i(x) = \frac{\partial f^i}{\partial x_j}(x)$ exist in an open set containing a and if each function $D_j f^i = \frac{\partial f^i}{\partial x_j}$ is continuous at a.

Remark: n HW 6 question 2 we looked at an example which shows that the condition that the partial derivatives are continuous cannot be removed. Namely, the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

We proved that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere on $\mathbb{R}^2$ and that the partial derivatives are not continuous at $(0,0)$. It is easy to see that f is **not** differentiable at $(0,0)$ as f is not continuous at $(0,0)$. (Remember that we proved in the homework that if a function is differentiable at a point, then it is continuous at that point.)

Conclusion: If a function is continuously differentiable at a point, then Theorem 2 proves that it is differentiable at that point. If a function has partial derivatives that exist but are not continuous at a point, then we don't have enough information to decide if the function is differentiable at that point.

2 Inverse Function Theorem

Theorem 3. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing a and $\det Df(a) \neq 0$. Then there is an open set V containing a and an open set W containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} W \to V$ which is differentiable and for all $y \in W$ satisfies*

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}.$$

That is, the Jacobian matrix of $Df^{-1}(y)$ is the inverse of the matrix of $Df(f^{-1}(y))]^{-1}$.

Remark: For $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = y$, this theorem becomes the familiar formula from Calculus

$$\frac{\partial x}{\partial y} = \frac{1}{\frac{\partial y}{\partial x}}.$$

Example: In class, we proved that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(u, v) = (2 - u + v^2, uv + 5)$ had an inverse at all points $\{(u, v) \in \mathbb{R}^2 \mid u \neq 2v^2\}$. We took the point $f(2, 3) = (9, 11)$ and used the inverse theorem to conclude that

$$Df^{-1}(9, 11) = [Df(f^{-1}(9, 11))]^{-1} = [Df(2, 3)]^{-1} = \begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{1}{20}\begin{bmatrix} -2 & 6 \\ 3 & 1 \end{bmatrix}.$$

Remark: The requirement in the Inverse Function Theorem that the function be continuously differentiable at the point in question cannot be removed. In HW 6 question 4 we considered the function

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and proved that f is differentiable at 0 and that $f'(0) = \frac{1}{2}$. We then proved that

$$f'(x) = \begin{cases} \frac{1}{2} + 2x \sin(\frac{1}{x})) - \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

and noted that $\lim_{x \to 0} f'(x)$ does not exist (as $\lim_{x \to 0} \cos(\frac{1}{x})$ does not exist). Hence $f'(x)$ is not continuous at 0. Finally we showed that there is no open interval containing 0 on which f is one-to-one. So having derivatives exist is not sufficient to to conclude there is a neighborhood about that point where the function is one to one. The Inverse function theorem requires the function to be continuously differentiable at the point in question.

3 Implicit Function Theorem

Theorem 4. *Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable in an open set containing (a, b) and $f(a, b) = 0$. Let M be the $m \times m$ matrix whose ijth entry is $D_{n+j} f^i(a, b) = \frac{\partial f^i}{\partial x_{n+j}}(a, b)$ for $1 \leq i, j, \leq m$. If $\det M \neq 0$, there is an open set $A \subset \mathbb{R}^n$ containing a and an open set $B \subset \mathbb{R}^m$ containing b, with the following property: for each $x \in A$, there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. The function g is differentiable.*