Math 225 Midterm Exam 2

Solutions

1. (3+3=6 points) Find the Jacobian matrix Df for the following functions.

   (a) $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(u,v,w) = (u^2vw - \sin u, (u+v)^4 - 5 + w^3)$

$$Df(u,v,w) = \begin{bmatrix} 2uvw - \cos u & u^2v & u^2w \\ 4(u+v)^3 & 4(u+v)^3 & 3w^2 \end{bmatrix}$$

   (b) $f : \mathbb{R} \to \mathbb{R}^3$ given by $f(t) = (2\cos t, 4\sin t, t^3 - \pi)$.

$$Df(t) = \begin{bmatrix} -2\sin t \\ 4\cos t \\ 3t^2 \end{bmatrix}$$

2. (2+4+3+0+3=12 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

   (a) Find $\frac{\partial f}{\partial x}$ when $(x,y) \neq (0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \frac{y(x^2+y^2) - 2x(xy)}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

   (b) Find $\frac{\partial f}{\partial x}$ when $(x,y) = (0,0)$. (Hint: use the definition of a partial derivative.)

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{0}{h^2+0} - 0}{h} \\ &= \lim_{h \to 0} \frac{0}{h} = 0 \end{aligned}$$

(c) For what points $(x, y) \in \mathbb{R}^2$ is $\frac{\partial f}{\partial x}$ continuous? (Explain your answer.)

Answer: $\frac{\partial f}{\partial x}$ is continuous for those (a, b) for which

$$\lim_{(x,y)\to(a,b)} \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial x}(a, b).$$

For $(a, b) \neq (0, 0)$, $\frac{\partial f}{\partial x}$ is a quotient of polynomial functions with nonzero denominator, so is continuous. For $(a, b) = (0, 0)$, consider approaching $(0, 0)$ along two different paths. First, the path $y = 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, 0) = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = \lim_{(x,0)\to(0,0)} 0 = 0.$$

However, approaching $(0, 0)$ along the path $y = x^2$ gives

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x^4 - x^2)}{(x^2 + x^4)^2} = \lim_{(x,y)\to(0,0)} \frac{x^4(x^2 - 1)}{x^4(1 + x^2)^2} = -1 \neq 0$$

Hence $\frac{\partial f}{\partial x}$ is not continuous at $(0, 0)$.

(d) Note that similar arguments will find $\frac{\partial f}{\partial y}$ at all points in $\mathbb{R}^2$, and will find the points $(x, y) \in \mathbb{R}^2$ where $\frac{\partial f}{\partial y}$ is continuous. (There is nothing for you to do here, but you might be able to use this comment.)

(e) Find the points $(x, y) \in \mathbb{R}^2$ where f differentiable. (Explain your answer, clearly stating any theorems you use.)

Answer: Recall the theorem that states that: If all partial derivatives of a function f exists in an open set about a fixed point a and if each partial derivative is continuous at a, then a f is differentiable at a. In parts (c) and (d) we proved that these conditions are fulfilled for $(x, y) \neq (0, 0)$ and so the function is differentiable $(x, y) \neq (0, 0)$.

The contrapositive of this theorem is the following: If a function f is not differentiable at a point a, then either all partial derivatives of a function f do not exist an open set about a fixed point a or each partial derivative is not continuous at a. (Not that the negation of "and" is "or".)

The converse of this theorem is FALSE. Many of you tried to use this to prove that f is not differentiable at $(0, 0)$. If a partial derivative is not continuous at a point, we DO NOT KNOW if the function is differentiable or not. If we know a function is differentiable at a point, then all we know is that the partial derivatives exist, not whether they are continuous or not. So if a partial derivative exists, but is not continuous, then this tells us that we can't use the theorem stated above, we have to try something else. Note that Question 1 of HW 6 gives you two examples of functions that are differentiable at $(0, 0)$ but whose partial derivatives are not continuous at $(0,0)$. If we know a function is differentiable at a point, then all we

We thus need to use a separate argument to show that f is not differentiable at $(0,0)$. Namely, recall the theorem you proved in your HW: If f is differentiable at a, then f is continuous at a. The contrapositive of this theorem states that if f is not continuous at a, then f is not differentiable at a. It is easy to check that our function f is not continuous at $(0,0)$. Approach $(0,0)$ along the path $x=0$, then

$$\lim_{(x,y)\to(0,0)} \frac{0y}{0^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{y^2} = \lim_{(x,y)\to(0,0)} 0 = 0.$$

Approach $(0,0)$ along the path $x=y$, then

$$\lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \lim_{(x,y)\to(0,0)} \frac{1}{2} = \frac{1}{2} \neq 0.$$

3. (2+2+4=8 points) Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^3$.

   (a) For what values of x is f a one-to-one function? For these values, what is the inverse function $f^{-1}:\mathbb{R}\to\mathbb{R}$.

   Answer: f is 1-1 for each $x\in\mathbb{R}$. Proof. Assume $x_1^3=x_2^3$. Then $0=x_1^3-x_2^3$ so $0=(x_1-x_2)(x_1^2+x_1x_2+x_2^2)$. This is only true if $x_1-x_2=0$, or if $x_1=x_2$. (As $x_1^2+x_1x_2+x_2^2=0$ has no real roots.)

   As f is 1-1 everywhere is has an inverse $f^{-1}(y)=\sqrt[3]{y}$.

   (b) For what value(s) of x is $\det Df(x)=0$? (Here $Df(x)$ is the Jacobian matrix of f at x.)

   Answer: $Df(x)=\begin{bmatrix}3x^2\end{bmatrix}$, so $\det Df(x)=3x^2$ and $\det Df(x)=0$ if and only if $x=0$.

   (c) Why do your answers to (a) and (b) not contradict the Inverse Function Theorem?

   Answer: The InFT can only be applied at a point a when f is continuously differentiable in an open set containing a point a and when $\det Df(a)\neq 0$. Here f is continuously differentiable everywhere and $\det Df(x)\neq 0$ for all $x\neq 0$. At these points f has an inverse which is continuously differentiable ($\sqrt[3]{y}$).

   The InFT can not be applied at $x=0$. This does not mean that there is not an inverse, just that we can't use the InFT. Indeed there is an inverse! However we do know that the inverse in at 0 cannot be differentiable. (If it were, then this would give a contradiction to the InFT - see class notes and Spivak p. 39.) A calculation gives $(f^{-1})'(y)=\frac{1}{3}\,{}^{-2/}\!\sqrt[3]{y}=\frac{1}{3}\frac{1}{{}^{2/}\!\sqrt[3]{y}}$, which is undefined at $y=0$.

4. (7 points) Let $g : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $g(x, y, z) = 1 + x - y^2 - z^2$ and let $g^{-1}(0) = \{(x, y, z) \,|\, g(x, y, z) = 0\}$. Explain why $g^{-1}(0)$ is a smooth (that is, differentiable) surface in $\mathbb{R}^3$.

   Answer: The aim is to use the general Implicit Function Theorem. We first check that the conditions for the theorem are met. As

   $$Dg(x, y, x) = \begin{bmatrix} 1 & -2y & -2z \end{bmatrix},$$

   we see that all partial derivatives are continuous at all points, so f is continuously differentiable. The rank of Dg is 1. In particular $\frac{\partial f}{\partial x} \neq 0$ for all (x, y, z). Hence if we take a point (a, b, c) with $g(a, b, c) = 0$, then we may apply the general ImFT. Thus there is an open set $A \subset \mathbb{R}^3$ containing (a, b, c) and a differentiable function $h : A \to \mathbb{R}^3$ such that $f \circ h(x, y, z) = (y, z)$. In particular x is implicitly a differentiable function of y and z, we might have written $g(f(y, z), y, z) = 0$ for all (y, z) near (b, c). Thus in a neighborhood of (a, b, c), $g^{-1}(0)$ is a smooth surface.

   Note that in our case we can actually write f explicitly! $f(y, z) = x = y^2 + z^2 - 1$.

   Also note that the regular Implicit Function Theorem only gives part of $g^{-1}(0)$ as a smooth surface. Since $M = \left[\frac{\partial f}{\partial z}\right] = \left[-2z\right]$, we may only apply the theorem when $z \neq 0$. (This leaves out a part of the surface.)

5. (6 points) Let $f : [0, 3] \times [-1, 1] \to \mathbb{R}$ be defined by

   $$f(x, y) = \begin{cases} 2 & \text{if } x \leq \frac{3}{2} \\ 3 & \text{if } \frac{3}{2} < x < 2 \\ -1 & \text{if } x \geq 2. \end{cases}$$

   Is f integrable? If so, find $\int_{[0,3]\times[-1,1]} f$.

   Answer: Note that f is a function of 2 variables, so $z = f(x, y)$. This question is not too different from HW 8 question 2.

   First pick $\epsilon > 0$. Second partition $[0, 3] \times [-1, 1]$ into sub-rectangles. We need to separate out the places where f is discontinuous, namely the line segments $x = \frac{3}{2}$ and $x = 2$. Let's take 5 subrectangles and work out $m_S(f)$ and $M_S(f)$ for each.
   $S_1 = [0, 3/2 - \epsilon/21] \times [-1, 1]$, $m_{S_1}(f) = 2 = M_{S_1}(f)$.
   $S_2 = [3/2 - \epsilon/21, 3/2 + \epsilon/21] \times [-1, 1]$, $m_{S_1}(f) = 2$ and $M_{S_1}(f) = 3$.
   $S_3 = [3/2 + \epsilon/21, 2 - \epsilon/21] \times [-1, 1]$, $m_{S_1}(f) = 3 = M_{S_1}(f)$
   $S_4 = [2 - \epsilon/21, 2 + \epsilon/21] \times [-1, 1]$, $m_{S_1}(f) = -1$ and $M_{S_1}(f) = 3$.
   $S_5 = [2 + \epsilon/21, 3] \times [-1, 1]$, $m_{S_1}(f) = -1 = M_{S_1}(f)$.

   The lower sum is

   $$\begin{aligned} L(f, P) &= 2(3/2 - \epsilon/21)2 + 2(2\epsilon/21)2 + 3(1/2 - 2\epsilon/21)2 + (-1)(2\epsilon/21)2 + (-1)(1 - \epsilon/21)2 \\ &= 6 - 4\epsilon/21 + 8\epsilon/21 + 3 - 12\epsilon/21 - 4\epsilon/21 - 1 + 2\epsilon/21 \\ &= 7 - 10\epsilon/21 \end{aligned}$$

The upper sum is

$$\begin{aligned}U(f,P) &= 2(3/2-\epsilon/21)2+3(2\epsilon/21)2+3(1/2-2\epsilon/21)2+(3)(2\epsilon/21)2+(-1)(1-\epsilon/21)2\\ &= 6-4\epsilon/21+12\epsilon/21+3-12\epsilon/21+12\epsilon/21-1+2\epsilon/21\\ &= 7+10\epsilon/21\end{aligned}$$

Third, $U(f,P)-L(f.P)=20\epsilon/21<\epsilon$, hence f is integrable.

Finally, for all $\epsilon>0$,

$$7-10\epsilon/21 \leq \sup\{L(f,P)\} \leq \inf\{U(f,P)\} \leq 7+10\epsilon/21.$$

Thus $\int_{[0,3]\times[-1,1]} f = \sup\{L(f,P)\} = \inf\{U(f,P)\} = 7$.

6. (3+3=6 points)

   (a) Let $B=(-\sqrt{2},4]\subset\mathbb{R}$. Does B have measure 0? Why or why not?

   Answer: B does not have measure 0.

   Take any cover $U_1, U_2, \ldots$ of B by closed intervals. As $B\subseteq\bigcup_i U_i$ and $v(U)=4+\sqrt{2}$, we know $\sum_i v(U_i)\geq 4+\sqrt{2}$. Pick $\epsilon=2$, then every cover of B has $\sum_i v(U_i)>2$.

   (b) Let $C=\{(x,y,z)\,|\,x\in[0,3], y\in[0,3], z=2\}\subset\mathbb{R}^3$. (So C is a square piece of the plane $z=2$ in $\mathbb{R}^3$.) Does C have measure 0? Why or why not?

   Answer: C has measure 0.

   Pick $\epsilon>0$. Take a cover of C to be the single rectangle

   $$U=[0,3]\times[0,3]\times[2-\epsilon/20, 2+\epsilon/20].$$

   Then $v(U)=(3)(3)(2\epsilon/20)=18\epsilon/20<\epsilon$, hence C has measure 0.