Group 2 Answers: Implicit Function theorem

1A) Let $f: \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ and define
$f^1: x^3 + yz - 2 = 0$
$f^2: xy + 2yz - z^2 - 2 = 0$
and at the point $f(1,1,1) = (f^1(1,1,1), f^2(1,1,1)) = (0,0)$.

Consider $Df = \begin{bmatrix} 3x^2 & z & y \\ y & x+2z & 2y-2z \end{bmatrix}$

then $M = \begin{bmatrix} z & y \\ x+2z & 2y-2z \end{bmatrix}$ (2x2 matrix)

at $(1,1,1)$ $M = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}$, thus $\det M = -3 \neq 0$.

Therefore, by IFT, $\exists$ an open set $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^2$, $(1) \in U$ and $(1,1) \in V$ s.t. $\forall x \in U$ $\exists!$ differentiable function $h(x,y,z) \in V$ and $f(x, h(x,y,z)) = 0$.
Equivalently in the neighborhood of $(1,1,1)$ $\exists$ a differentiable function $h: \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $h(x,y,z) = (y,z)$.

1B) Since $h(x,y,z) = (y,z)$, we can write $h(x,y,z) = (h^1(x,y,z), h^2(x,y,z))$. Then, Df is 2x3 matrix: $\begin{bmatrix} \frac{\partial h^1}{\partial x} & \frac{\partial h^1}{\partial y} & \frac{\partial h^1}{\partial z} \\ \frac{\partial h^2}{\partial x} & \frac{\partial h^2}{\partial y} & \frac{\partial h^2}{\partial z} \end{bmatrix}$

Find $\frac{\partial f^1}{\partial x}$ by taking derivatives of $0 = f(x, h(x,y,z))$

$\rightarrow$

Therefore,

$$\star\ Dh(1,1,1) = \begin{bmatrix} -1/3 & 0 & 0 \\ -2\,2/3 & 0 & 0 \end{bmatrix}.$$

$$\frac{\partial f^1}{\partial x}: \frac{\partial f^1}{\partial x}\overset{\Rightarrow 1}{\cancel{\frac{\partial x}{\partial x}}} + \frac{\partial f^1}{\partial y}\overset{\Rightarrow \frac{\partial h^1}{\partial x}}{\left(\frac{\partial y}{\partial x}\right)} + \frac{\partial f^1}{\partial z}\overset{\Rightarrow \frac{\partial h^2}{\partial x}}{\left(\frac{\partial z}{\partial x}\right)}$$

at $(1,1,1) \Rightarrow$ $$= 3x^2 + 2\left(\frac{\partial h^1}{\partial x}\right) + y\left(\frac{\partial h^2}{\partial x}\right) = 3 + \frac{\partial h^1}{\partial x} + \frac{\partial h^2}{\partial x} = -2\,2/3$$

$$\frac{\partial f^2}{\partial x}: \frac{\partial f^2}{\partial x}\underbrace{\cancel{\left(\frac{\partial x}{\partial x}\right)}}_{=1} + \frac{\partial f^2}{\partial y}\overset{=\frac{\partial h^1}{\partial x}}{\left(\frac{\partial y}{\partial x}\right)} + \frac{\partial f^2}{\partial z}\left(\frac{\partial z}{\partial x}\right) \Leftarrow \frac{\partial h^2}{\partial x}$$

at $(1,1,1) \Rightarrow$ $$= y(1) + x+2z\left(\frac{\partial h^1}{\partial x}\right) + 2y-2z\left(\frac{\partial h^2}{\partial x}\right) = 1 + 3\left(\frac{\partial h^1}{\partial x}\right)$$ $$\Rightarrow \frac{\partial h^1}{\partial x} = \frac{-1}{3}$$

$$\boxed{\text{Thus } \frac{\partial h^1}{\partial x} = \frac{-1}{3} \text{ and } \frac{\partial h^2}{\partial x} = -2\,2/3}$$

$$\frac{\partial f^1}{\partial y}: \frac{\partial f^1}{\partial x}\overset{0}{\cancel{\left(\frac{\partial x}{\partial y}\right)}} + \frac{\partial f^1}{\partial y}\overset{=\frac{\partial h^1}{\partial y}}{\left(\frac{\partial y}{\partial y}\right)} + \frac{\partial f^1}{\partial z}\overset{=\frac{\partial h^2}{\partial y}}{\left(\frac{\partial z}{\partial y}\right)}$$

$$= 0 + z\left(\frac{\partial h^1}{\partial y}\right) + y\left(\frac{\partial h^2}{\partial y}\right) = \frac{\partial h^1}{\partial y} + \frac{\partial h^2}{\partial y} = 0$$

$$\frac{\partial f^2}{\partial y}: \frac{\partial f^2}{\partial x}\overset{0}{\cancel{\left(\frac{\partial x}{\partial y}\right)}} + \frac{\partial f^2}{\partial y}\left(\frac{\partial y}{\partial y}\right) = \frac{\partial h^1}{\partial y} + \frac{\partial f^2}{\partial z}\overset{=\frac{\partial h^2}{\partial y}}{\left(\frac{\partial z}{\partial y}\right)}$$

$$= 0 + x+2z\left(\frac{\partial h^1}{\partial y}\right) + 2y-2z\left(\frac{\partial h^2}{\partial y}\right) = \frac{3\partial h^1}{\partial y} = 0$$

$$\boxed{\text{Thus } \frac{\partial h^1}{\partial y} \text{ and } \frac{\partial h^2}{\partial y} \text{ both equal } 0.}$$

$$\frac{\partial f^1}{\partial z}: \frac{\partial f^1}{\partial x}\overset{=0}{\cancel{\left(\frac{\partial x}{\partial z}\right)}} + \frac{\partial f^1}{\partial y}\overset{=\frac{\partial h^1}{\partial z}}{\left(\frac{\partial y}{\partial z}\right)} + \frac{\partial f^1}{\partial z}\overset{=\frac{\partial h^2}{\partial z}}{\left(\frac{\partial z}{\partial z}\right)}$$

$$0 + z\left(\frac{\partial h^1}{\partial z}\right) + y\left(\frac{\partial h^2}{\partial z}\right) = \frac{\partial h^1}{\partial z} + \frac{\partial h^2}{\partial z} = 0$$

$$\frac{\partial f^2}{\partial z}: \frac{\partial f^2}{\partial x}\overset{=0}{\cancel{\left(\frac{\partial x}{\partial z}\right)}} + \frac{\partial f^2}{\partial y}\overset{=\frac{\partial h^1}{\partial z}}{\left(\frac{\partial y}{\partial z}\right)} + \frac{\partial f^2}{\partial z}\overset{=\frac{\partial h^2}{\partial z}}{\cancel{\left(\frac{\partial z}{\partial z}\right)}}$$

$$= 0 + x+2z\left(\frac{\partial h^1}{\partial z}\right) + 2y-2z\left(\frac{\partial h^2}{\partial z}\right) = 3\frac{\partial h^1}{\partial z} = 0$$

$$\boxed{\text{Thus } \frac{\partial h^1}{\partial z} \text{ and } \frac{\partial h^2}{\partial z} \text{ both equal } 0.}\ \star$$

(2) $f: \mathbb{R}^3 \to \mathbb{R}$

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} (y+e^x) & (x+e^x+z\sin y) & -\cos y \end{bmatrix}$$

$$M = \begin{bmatrix} -\cos y \end{bmatrix}$$

$\det(M) = 0$ when $(x, y, z) = (x, \frac{\pi}{2} + \pi n, z)$

$\det(M) \neq 0$ when $(x, y, z) \neq (x, \frac{\pi}{2} + \pi n, z)$

Thus, we can apply the implicit function Theorem $\forall (x, y, z) \in \mathbb{R}^3$ such that $(x, y, z) \neq (x, \frac{\pi}{2} + \pi n, z)$ and
$f(x, y, z) = 0$.

We can't apply IFT $\forall (x, y, z) \in \mathbb{R}^3$ such that
$(x, y, z) = (x, \pi/2 + \pi n, z)$ or
$f(x, y, z) \neq 0$.

③

(a) Let $\vec{e}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad \vec{e}_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\quad \vec{e}_3 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Note: $X^1 = X_1$ the superscript changed to subscript for convenience

Then $A = \begin{bmatrix} - & \vec{e}_1 & - \\ - & e_2 & - \\ - & e_3 & - \end{bmatrix}$ and $A^T = \begin{bmatrix} | & | & | \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ | & | & | \end{bmatrix}$

So $AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ label as $\begin{bmatrix} ① & ② & ③ \\ ④ & ⑤ & ⑥ \\ ⑦ & ⑧ & ⑨ \end{bmatrix}$

$O(3)$ is the solution set of 6 equations. Note that $\vec{e}_i$ is orthonormal basis.

The equations are:

for ① $\vec{e}_1 \cdot \vec{e}_1 = \|\vec{e}_1\|^2 = 1^2 = 1$

② and ④ $\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_2 \cdot \vec{e}_1 = 0$

③ and ⑦ $\vec{e}_1 \cdot \vec{e}_3 = \vec{e}_3 \cdot \vec{e}_1 = 0$

⑤ $\vec{e}_2 \cdot \vec{e}_2 = 1$

⑥ and ⑧ $\vec{e}_2 \cdot \vec{e}_3 = \vec{e}_3 \cdot \vec{e}_2 = 0$

⑨ $\vec{e}_3 \cdot \vec{e}_3 = 1$

$$\boxed{\begin{aligned} f_1 &= x_1^2 + x_2^2 + x_3^2 - 1 \\ f_2 &= x_1 y_1 + x_2 y_2 + x_3 y_3 \\ f_3 &= x_1 z_1 + x_2 z_2 + x_3 z_3 \\ f_4 &= y_1^2 + y_2^2 + y_3^2 - 1 \\ f_5 &= y_1 z_1 + y_2 z_2 + y_3 z_3 \\ f_6 &= z_1^2 + z_2^2 + z_3^2 \end{aligned}}$$

(b) Dg is 6×9 matrix s.t $\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial z_3} \\ \vdots & & \\ \frac{\partial f_6}{\partial x_1} & \cdots & \frac{\partial f_6}{\partial z_3} \end{bmatrix}$

$$= \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & 0 & 0 & 0 \\ z_1 & z_2 & z_3 & 0 & 0 & 0 & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & 2y_1 & 2y_2 & 2y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1 & z_2 & z_3 & y_1 & y_2 & y_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2z_1 & 2z_2 & 2z_3 \end{bmatrix} \begin{array}{l} \leftarrow r_1 \\ \leftarrow r_2 \\ \\ \\ \\ \leftarrow r_3 \end{array}$$

(c) I show Dg has full rank of 6 because all the rows in Dg are pairwise perpendicular.

Consider the rows r_1, r_2, r_3 in Dg.

$$\vec{r_1} \cdot \vec{r_2} = 2x_1y_1 + 2x_2y_2 + 2x_3y_3 + 0x_1 + 0x_2 + 0x_3 + 0 + 0 + 0$$

$$= 2\sum_{i=1}^{3} x_i y_i \qquad \text{here; } \sum_{i=1}^{3} x_i y_i = 0 \text{ since } \vec{e_i} \text{ is orthonormal, So}$$

$$= 2(0) = 0 \qquad \therefore \vec{r_1} \perp \vec{r_2}$$

$$\vec{r_2} \cdot \vec{r_3} = 0y_1 + 0y_2 + 0y_3 + 0x_1 + 0x_2 + 0x_3 + 0z_1 + 0z_2 + 0z_3 = 0$$

$$\therefore \vec{r_2} \perp \vec{r_3}$$

Similarly, all the rows are pairwise perpendicular, so none of row can be written as a linear combination of other rows.

Therefore, Dg must have full rank of 6.

Then by gIFT, we can find neighborhood of A in where we can write 3 variables as differential functions of the other 6 variables.

Thus, $O(3)$ is a smooth submanifold of $\mathbb{R}^9$ $\blacksquare$

(d) $A = I$ so $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then

$$Dg = \begin{array}{c} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & z_1 & z_2 & z_3 \end{matrix} \end{array}$$

The columns corresponding to y_1, z_1, z_2 linearly dependent to other columns.

Thus, y_1, z_1, z_2 can be written as a differentiable function of the other variables.