Math 225 Homework 9

Deadline between 4pm Friday April 10 and 3pm Tuesday April 14, 2009.

1. (a) *Volume and linear change of variables*

   Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of one of the following types:

   i. $\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = ae_j \end{cases}$

   For example consider the matrix $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   ii. $\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = e_j + e_k \end{cases}$

   For example consider the matrix $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   iii. $\begin{cases} g(e_k) = e_k & k \neq i, j \\ g(e_i) = e_j \\ g(e_j) = e_i \end{cases}$

   For example consider the matrix $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

   If U is a rectangle, show that in each case the volume of $g(U)$ is $|\det(g)| \cdot vol(U)$. To get started on this, use $U = [0, 1] \times [0, 1] \times [0, 1]$ and the matrices E_1, E_2, E_3. Then prove the general case.

   (b) Prove that $|\det(g)| \cdot vol(U)$ is the volume of $g(U)$ for **any** linear transformation $g : \mathbb{R}^n \to \mathbb{R}^n$. (Hint: There are two cases. If $\det g \neq 0$, then g is the composition of linear transformations of the type considered in part (a). Now take $\det g = 0$.)

2. *Measure* 0

   In class, we discuss the definition of measure 0 subsets of $\mathbb{R}^n$. Use the arguments given in class to do the following problems:

   (a) Prove that $[0, 1] \subset \mathbb{R}$ does not have measure 0.

(b) Prove that $A = \{1, 2, 3, 4, 5\} \subset \mathbb{R}$ has measure 0.

(c) Prove that $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \subset \mathbb{R}$ has measure 0.

(d) Given $A \subset [0, 1] \times [0, 1]$ and A is the line segment $x = \frac{1}{2}$ and $0 \le y \le 1$, prove that A has measure 0.

3. *Integration review*

(a) Calculate the iterated integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \sin(x + y)\, dy\, dx$$

(b) Evaluate the iterated integral by first changing the order of integration. (Hint: first draw a picture of the region of integration.)

$$\int_0^3 \int_{y^2}^9 y \cos(x^2)\, dx\, dy$$

(c) Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2 + 4$ and the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$.

4. *More integration*

(a) Use polar coordinates to find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \le 9$.

(b) Use cylindrical coordinates to evaluate $\iiint_E \sqrt{x^2 + y^2}\, dV$, where E is the region inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

(c) Use spherical coordinates to evaluate $\iiint_E z\, dV$, where E is the solid that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.