Math 225 Homework 8

Due 4pm on Friday April 3, 2009.

1. *Supremum and infimum of $A \subset \mathbb{R}$.*

   (a) Prove that the infimum and supremum of a set $A \subset \mathbb{R}$ are both unique.

   (b) Let $A = \{\frac{1}{n} \,|\, n = 1, 2, 3, \dots\}$. Prove (using the definition) that $\inf(A) = 0$.

   (c) Let $A \subset \mathbb{R}$. Define the "negative" of A to be the set $-A = \{-a \,|\, a \in A\}$. Prove that

   $$-\sup(A) = \inf(-A).$$

2. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

   $$f(x,y) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

   Show (using the definition from class) that f is integrable and $\int_{[0,1]\times[0,1]} f = \frac{1}{2}$.

3. Let $A \subset \mathbb{R}^n$ be a rectangle and $f, g : A \to \mathbb{R}$ be integrable.

   (a) For any partition P of A and subrectangle S, show that
   $m_s(f) + m_s(g) \leq m_s(f+g)$ and $M_s(f+g) \leq M_s(f) + M_s(g)$,
   and therefore
   $L(f,P) + L(g,P) \leq L(f+g,P)$ and $U(f+g,P) \leq U(f,P) + U(g,P)$.

   (b) Show that $f+g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.

   (c) For any constant c, show that $\int_A (cf) = c \int_A f$

4. Let $A \subset \mathbb{R}^n$ be a rectangle, $f, g : A \to \mathbb{R}$ be integrable and suppose that $f \leq g$. Show that $\int_A f \leq \int_A g$.

5. *Determinant of a 2×2 matrix.*

   Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and consider column vectors $\vec{v} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} b \\ d \end{bmatrix}$. Also consider $\vec{v}_{rot} = \begin{bmatrix} -c \\ a \end{bmatrix}$, obtained by rotating vector $\vec{v}$ by $\pi/2$ in a counter-clockwise direction. Let θ be the angle oriented from $\vec{v}$ to $\vec{w}$. (Hint: Confused? Draw a picture!)

(a) Compute $\det A$ and $\vec{v}_{rot} \cdot \vec{w}$. What do you notice?

(b) Observe that $\vec{v}_{rot} \cdot \vec{w} = ||\vec{v}_{rot}||\,||\vec{w}||\cos(\pi/2 - \theta) = ||\vec{v}||\,||\vec{w}||\sin\theta$. Why is the last equality true?

(c) Use geometry to show that the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$ is $||\vec{v}||\,||\vec{w}||\,|\sin\theta|$.

(d) Combining parts (a),(b) and (c) we see that $|\det A| = ||\vec{v}||\,||\vec{w}||\,|\sin\theta|$ is *the area of the parallelogram spanned by* $\vec{v}$ *and* $\vec{w}$.

When is $\det A = 0$? When is $\det A > 0$? When is $\det A < 0$? Please draw a picture illustrating each case.