Math 225 Homework 7

Solutions.

In this HW set we use the following two theorems:

Theorem 1 (Implicit Function Theorem)**.** *Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable in an open set containing (a, b) and $f(a, b) = 0$. Let M be the $m \times m$ matrix whose ijth entry is $D_{n+j}f^i(a, b) = \frac{\partial f^i}{\partial x_{n+j}}(a, b)$ for $1 \leq i, j, \leq m$. If $\det M \neq 0$, there is an open set $A \subset \mathbb{R}^n$ containing a and an open set $B \subset \mathbb{R}^m$ containing b, with the following property: for each $x \in A$, there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. The function g is differentiable.*

Note that if $f(a, b) = k$ for a constant k (rather than 0), then the theorem may still be applied. (Let $\bar{f} = f - k$. Then $\bar{f} = 0$ if and only if $f = k$. Now apply the IFT to $\bar{f}$.)

Theorem 2 (General Implicit Function Theorem)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^p$ is continuously differentiable in an open set containing a, where $p \leq n$. If $f(a) = 0$ and the $p \times n$ Jacobian matrix $(D_j f^i(a, b) = \frac{\partial f^i}{\partial x_j}(a))$ has rank p, then there is an open set $A \subset \mathbb{R}^n$ containing a and a differentiable function $h : A \to \mathbb{R}^n$ with differentiable inverse such that*

$$f \circ h(x_1, \ldots, x_n) = (x_{n-p+1}, \ldots, x_n).$$

This last theorem means that in a neighborhood of a, p variables are a differentiable function of $n - p$ variables.

1. *Level surfaces*

   (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose that $f(a, b, c) = k$, and that

   $$\frac{\partial f}{\partial z}(a, b, c) \neq 0.$$

   Show that there is an open set $U \subset \mathbb{R}^2$ containing (a, b) and a differentiable function $g : U \to \mathbb{R}$ such that

   $$f(a, b, g(a, b)) = k.$$

   Explain why the points in the set

   $$\{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = k\}$$

which are sufficiently close to (a, b, c) form a smooth surface in $\mathbb{R}^3$.

Answer: First note that f is smooth (which means it is continuously differentiable) and note $f(a, b, c) = k$. Now $Df(a, b, c) = \begin{bmatrix} \frac{\partial f(a,b,c)}{\partial x} & \frac{\partial f(a,b,c)}{\partial y} & \frac{\partial f(a,b,c)}{\partial z} \end{bmatrix}$. Hence $\det M = \det \begin{bmatrix} \frac{\partial f(a,b,c)}{\partial z} \end{bmatrix} \neq 0$ by assumption.

This means we can use the implicit function theorem. Thus there is an open set $U = A \times B$ of (a, b, c), where $A \subset \mathbb{R}^2$ and $(a, b) \in A$ and $c \in B \subset \mathbb{R}$ with the following property: For each $(x, y) \in A$, there is a unique $g(x, y) = z \in B$ such that $f(x, y, g(x, y)) = k$. Moreover g is differentiable.

Thus we may conclude that $f(a, b, g(a, b)) = k$. Also all the points sufficiently close to (a, b, c), that is for $(x, y, z) \in A \times B$, we can write z as a smooth function of x and y. This is a smooth surface in $\mathbb{R}^3$.

(b) Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and that $k \in \mathbb{R}$ is such that the partial derivatives

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z}$$

do not simultaneously vanish anywhere on the set

$$f^{-1}(k) = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = k\}.$$

Explain why the set $f^{-1}(k)$ forms a smooth surface in R^3. $f^{-1}(k)$ is called the level surface of f at level k. *You need to show that near each point $(a, b, c) \in f^{-1}(k)$, the set $f^{-1}(k)$ looks like a smooth surface in $\mathbb{R}^3$.*

Answer: We want to apply the general Implicit Function Theorem. Note that f is continuously differentiable and pick a point $(a, b, c) \in f^{-1}(k)$. The Jacobian $Df(a, b, c) = \begin{bmatrix} \frac{\partial f(a,b,c)}{\partial x} & \frac{\partial f(a,b,c)}{\partial y} & \frac{\partial f(a,b,c)}{\partial z} \end{bmatrix}$. We are given that the partial derivatives do not vanish simultaneously at (a, b, c), thus $\text{rank}(Df(a, b, c)) = 1$. Without loss of generality, suppose that $\frac{\partial f(a,b,c)}{\partial y} \neq 1$.

Apply the general IFT, there is an open set $A \subset \mathbb{R}^3$ with $(a, b, c) \in A$ and a differentiable $h : A \to \mathbb{R}^3$ such that for $(x, y, z) \in A$, $f \circ h(x, y, z) = (x, z)$. This means that in A we may write y as a differentiable function of x and z. (If $\frac{\partial f(a,b,c)}{\partial x} \neq 1$, we may write x as a differentiable function of y and z, etc.) This means that sufficiently close to the point (a, b, c) $f^{-1}(k)$ is a smooth surface. This argument holds for any point in $f^{-1}(k)$, hence it is a smooth surface in $\mathbb{R}^3$.

2. Consider the set S of points in $\mathbb{R}^5$ defined by the two equations:

$$xu^2 + yzv + x^2z = 3$$

$$xyv^3 + 2zu - u^2v^2 = 2$$

(a) Show there is a neighborhood of the point $(1,1,1,1,1) \in S$ and a differentiable function $h : \mathbb{R}^3 \to \mathbb{R}^2$ such that in the neighborhood, the point $(x,y,z,u,v) \in S$ where $h(x,y,z) = (u,v)$.

Answer: Let $f : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by $f(x,y,z,u,v) = (f^1, f^2)$, where $f^1(x,y,z,u,v) = xv^2 + yzv + x^2z - 3$ and $f^2(x,y,z,u,v) = xyv^3 + 2zu - u^2v^2 - 2$. Note that f is continuously differentiable and $f(1,1,1,1,1) = (1+1+1-3, 1+2-1-2) = (0,0)$. The Jacobian is

$$Df(x,y,z,u,v) = \begin{bmatrix} u^2 + 2xz & zv & yv + x^2 & 2xu & yz \\ yv^3 & yv^3 & 2v & 2z - 2uv^2 & 3xyv^2 - 2u^2v \end{bmatrix},$$

and at $(1,1,1,1,1)$

$$Df(1,1,1,1,1) = \begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 0 & 1 \end{bmatrix}.$$

Then $M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, so $\det M = 2 \neq 0$.

Thus, we may apply the Implicit Function Theorem. There is an open set $A \subset \mathbb{R}^3$ and $(1,1,1) \in A$, an open set $B \subset \mathbb{R}^2$ with $(1,1) \in B$ and a differentiable function $h : A \to B$ such that for each $(x,y,z) \in A$, $f(x,y,z,h(x,y,z)) = (0,0)$, or $(u,v) = h(x,y,z)$.

(b) Find $Dh(1,1,1)$. (Find the matrix of the Jacobian matrix of h at $(1,1,1)$.)

Answer: Note that $h : \mathbb{R}^3 \to \mathbb{R}^2$, hence

$$Dh = \begin{bmatrix} \frac{\partial h^1}{\partial x} & \frac{\partial h^1}{\partial y} & \frac{\partial h^1}{\partial z} \\ \frac{\partial h^2}{\partial x} & \frac{\partial h^2}{\partial y} & \frac{\partial h^2}{\partial z} \end{bmatrix}.$$

The best way to find the partial derivatives is to use the Chain rule. As $f(x,y,z,h(x,y,z)) = (0,0)$, we know $f^1(x,y,z,h^1(x,y,z),h^2(x,y,z)) = 0$ and $f^2(x,y,z,h^1(x,y,z),h^2(x,y,z)) = 0$ and that all partial derivatives are 0. Then

$$\begin{aligned} 0 = \frac{\partial f^1}{\partial x} &= \frac{\partial f^1}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f^1}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f^1}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f^1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f^1}{\partial v}\frac{\partial v}{\partial x} \\ &= \frac{\partial f^1}{\partial x}1 + \frac{\partial f^1}{\partial y}0 + \frac{\partial f^1}{\partial z}0 + \frac{\partial f^1}{\partial u}\frac{\partial h^1}{\partial x} + \frac{\partial f^1}{\partial v}\frac{\partial h^2}{\partial x} \\ &= (u^2 + 2xz) + 2xu\frac{\partial h^1}{\partial x} + yz\frac{\partial h^2}{\partial x} \end{aligned}$$

Hence $\frac{\partial f^1}{\partial x}(1,1,1) = 3 + 2\frac{\partial h^1}{\partial x} + 1\frac{\partial h^2}{\partial x}$. Now we could proceed and compute each of these, then solve for the partials of h. This is what everybody did, some using substitution, some using matrices.

We could also take the following short cut. In the calculation above, we wrote $f(x, y, z, h^1(x, y, z), h^2(x, y, z)) = (0, 0)$, then used Chain rule to compute. Note that we could rewrite this as $f(H(x, y, z)) = 0$, where $H^1(x, y, z) = x$, $H^2(x, y, z) = y$, $H^3(x, y, z) = z$, $H^4(x, y, z) = h^1(x, y, z)$ and $H^5(x, y, z) = h^2(x, y, z)$. The chain rule states $D(f \circ H)(1, 1, 1) = Df(H(1, 1, 1))DH(1, 1, 1)$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial h^1}{\partial x} & \frac{\partial h^1}{\partial y} & \frac{\partial h^1}{\partial z} \\ \frac{\partial h^2}{\partial x} & \frac{\partial h^2}{\partial y} & \frac{\partial h^2}{\partial z} \end{bmatrix}.$$

Rewriting this gives

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial h^1}{\partial x} & \frac{\partial h^1}{\partial y} & \frac{\partial h^1}{\partial z} \\ \frac{\partial h^2}{\partial x} & \frac{\partial h^2}{\partial y} & \frac{\partial h^2}{\partial z} \end{bmatrix},$$

or

$$\begin{aligned} \begin{bmatrix} \frac{\partial h^2}{\partial x} & \frac{\partial h^2}{\partial y} & \frac{\partial h^2}{\partial z} \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}^{-1} (-1) \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & -1 & -2 \\ -1 & -1 & -2 \end{bmatrix} \\ &= \frac{1}{2} = \begin{bmatrix} -2 & 0 & 0 \\ -2 & -2 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & -2 \end{bmatrix} \end{aligned}$$

3. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be given by $F(x, y, z) = x^2 + y^2 - z^2$. What values does the rank of DF take at different points of $\mathbb{R}^3$? Sketch $F^{-1}(a)$ for $a = -1, 0, 1$. Which of these surfaces (in $\mathbb{R}^3$) is smooth at every point? Give a brief (sentence or two) explanation why. (This question should help give a bit of insight into the general Implicit Function Theorem.)

   Answer First note that F is continuously differentiable and that $DF = \begin{bmatrix} 2x & 2y & -2z \end{bmatrix}$. $DF(0, 0, 0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ which has rank 0. All other values of (x, y, z) have one nonzero term, which means $\text{rank}(Df) = 1$. We are asked to consider $F^{-1}(-1)$, $F^{-1}(0)$ and $F^{-1}(1)$.

The best pictures of these surfaces can be found in Section 9.6 of Stewart's text Multivariable Calculus concepts and contexts 3rd ed. (It's in my office if you'd like to look at it.)

Now $F^{-1}(-1) = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1\}$. This is a hyperboloid of two sheets, with axis the z-axis. Note that $(0,0,0) \notin F^{-1}(-1)$ and so all points of $F^{-1}(-1)$ have $\text{rank}(Df) = 1$. We may apply question 1b to conclude that this is a smooth surface in $\mathbb{R}^3$, as the pictures indicate.

Similarly $F^{-1}(1) = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$. This is a hyperboloid of one sheets, with axis the z-axis. Note that $(0,0,0) \notin F^{-1}(-1)$ and so all points of $F^{-1}(-1)$ have $\text{rank}(Df) = 1$. We may apply question 1b to conclude that this is a smooth surface in $\mathbb{R}^3$, as the pictures indicate.

Finally $F^{-1}(0) = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0\}$ is a cone, and $F(0,0,0) = 0$. Hence $\text{rank}(Df(0,0,0)) = 0$ and $\text{rank}(DF) = 1$ at all other points of $F^{-1}(0)$. Thus at $(0,0,0)$, the cone is not smooth, but away from $(0,0,0)$, the cone is a smooth surface. This is exactly what we see in the picture.

4. $O(n) =$ "the orthogonal group" $=$ all choices of orthonormal basis $\{\vec{e_1}, \ldots, \vec{e_n}\}$ of $\mathbb{R}^n$ $=$ all $n \times n$ real matrices A such that $AA^T = I$. Here

$$A = \begin{pmatrix} \vec{e_1} \\ \vdots \\ \vec{e_n} \end{pmatrix}$$

(a) $O(4)$ is the solution set of 10 equations in the variables

$$(x^1, x^2, x^3, x^4, y^1, y^2, y^3, y^4, z^1, z^2, z^3, z^4, w^1, w^2, w^3, w^4).$$

Write down these equations. (Note: you can also write these variables as $(\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4})$ where $\vec{e_i} \in \mathbb{R}^4$, which make make this a little easier.)

Answer: One way to think about $O(4)$ is that it is the number of distinct orthonormal bases of $\mathbb{R}^4$. Let an othonormal basis for $\mathbb{R}^4$ be $\vec{e}_1 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix}$, $\vec{e}_4 = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}$.
The 10 equations come from

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } i, j = 1, 2, 3, 4.$$

Example: $\vec{e}_1 \cdot \vec{e}_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, and $\vec{e}_1 \cdot \vec{e}_2 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = 0$.

(b) Write down the 10×16 Jacobian matrix Dg:

Answer: I'll give the abbreviated version of the matrix

$$Dg = \begin{bmatrix} 2\vec{e}_1 & 0 & 0 & 0 \\ \vec{e}_2 & \vec{e}_1 & 0 & 0 \\ \vec{e}_3 & 0 & \vec{e}_1 & 0 \\ \vec{e}_4 & 0 & 0 & \vec{e}_1 \\ 0 & 2\vec{e}_2 & 0 & 0 \\ 0 & \vec{e}_3 & \vec{e}_2 & 0 \\ 0 & \vec{e}_4 & 0 & \vec{e}_2 \\ 0 & 0 & 2\vec{e}_3 & 0 \\ 0 & 0 & \vec{e}_4 & \vec{e}_3 \\ 0 & 0 & 0 & 2\vec{e}_4 \end{bmatrix}.$$

Here, there are 16 columns, not 4. Simply replace the $\vec{e}_1$ by x_1, x_2, x_3, x_4 etc.

(c) Show that $Dg(A)$ has full rank for every $A \in O(4)$. So, by the general Implicit Function Theorem, $O(4)$ is smooth submanifold of $\mathbb{R}^{16}$. (There is a really easy way to do the computation, and a not-too-hard way.)

Answer: $\operatorname{rank}(Dg(A)) = 10$ when the rows are linearly independent. This is certainly the case when they are pairwise orthogonal. So compute the dot product between all pairs of rows. They will all be zero, either because the dot product is zero, or because we use the equations from part (a). For example $\text{row}_1 \cdot \text{row}_2 = \begin{bmatrix} 2\vec{e}_1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \vec{e}_2 & \vec{e}_1 & 0 & 0 \end{bmatrix} = (2\vec{e}_1) \cdot \vec{e}_2 + 0 + 0 + 0 = 2(\vec{e}_1 \cdot \vec{e}_2) = 0$, or $\text{row}_4 \cdot \text{row}_5 = \begin{bmatrix} \vec{e}_4 & 0 & 0 & \vec{e}_1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2\vec{e}_2 & 0 & 0 \end{bmatrix} = 0 + 0 + 0 + 0 = 0$, or $\text{row}_7 \cdot \text{row}_{10} = \begin{bmatrix} 0 & \vec{e}_4 & 0 & \vec{e}_2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 2\vec{e}_4 \end{bmatrix} = 0 + 0 + 0 + \vec{e}_2 \cdot (2\vec{e}_4) = 0 + 0 + 0 + 2(\vec{e}_2 \cdot \vec{e}_4) = 0$.

Thus given an orthogonal matrix A, $\operatorname{rank}(Dg(A)) = 10$. Applying the general IFT, there is an open neighborhood U of A in $O(4) \cong \mathbb{R}^{16}$ and a differentiable function $h : U \to \mathbb{R}^{16}$ such that $g(h(x_1, \ldots, w_4)) = 6$ of the 16 variables, for example $(z_3, z_4, w_1, w_, w_3, w_4)$. This in turn means that these 6 variables can be written as a smooth function of the other 10 variables. This is true for each point/matrix in U. Thus in U, $O(4)$ is a smooth 6-dimensional "surface" in or submanifold of $\mathbb{R}^{16}$. We can repeat this argument for each orthogonal matrix, hence $O(4)$ is a smooth submanifold of $\mathbb{R}^{16}$.

(d) By the general Implicit Function Theorem, any $A \in O(4)$ is in an open set and some of the variables $(x^1, \ldots, w^4)$ may be written as a differentiable function of the other variables. Give such a choice for $A = I$.

Answer: Note that the question is just repeating the arguments above. If we let A_I, then $x_1 = 1 = y_2 = z_3 = w_4$ and all other terms are zero. Substitution yields

$$
Dg(I) = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
$$

Rank 10 is achieved by choosing columns: 1, 2 3, 4, 6, 7, 8, 11, 12, 16. This means that variables $y_1, z_1, z_2, w_1, w_2, w_3$ may be written as a differentiable function of the other 10 variables.

Another choice is columns: 1, 5, 9, 6, 7, 8, 11, 12, 13, 15, 16. This means that variables $x_2, x_3, x_4, z_2, z_4, w_2$ may be written as a differentiable function of the other 10 variables.