Math 225 Homework 7

Due 4pm on Fri March 13, 2009.

1. *Level surfaces*

   (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose that $f(a, b, c) = k$, and that

   $$\frac{\partial f}{\partial z}(a, b, c) \neq 0.$$

   Show that there is an open set $U \subset \mathbb{R}^2$ containing (a, b) and a differentiable function $g : U \to \mathbb{R}$ such that

   $$f(a, b, g(a, b)) = k.$$

   Explain why the points in the set

   $$\{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = k\}$$

   which are sufficiently close to (a, b, c) form a smooth surface in $\mathbb{R}^3$.

   (b) Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and that $k \in \mathbb{R}$ is such that the partial derivatives

   $$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z}$$

   do not simultaneously vanish anywhere on the set

   $$f^{-1}(k) = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = k\}.$$

   Explain why the set $f^{-1}(k)$ forms a smooth surface in R^3. $f^{-1}(k)$ is called the level surface of f at level k. *You need to show that near each point* $(a, b, c) \in f^{-1}(k)$*, the set* $f^{-1}(k)$ *looks like a smooth surface in* $\mathbb{R}^3$*.*

2. Consider the set S of points in $\mathbb{R}^5$ defined by the two equations:

   $$xu^2 + yzv + x^2 z = 3$$

   $$xyv^3 + 2zu - u^2 v^2 = 2$$

(a) Show there is a neighborhood of the point $(1, 1, 1, 1, 1) \in S$ and a differentiable function $h : \mathbb{R}^3 \to \mathbb{R}^2$ such that in the neighborhood, the point $(x, y, z, u, v) \in S$ where $h(x, y, z) = (u, v)$.

(b) Find $Dh(1, 1, 1)$. (Find the matrix of the Jacobian matrix of h at $(1, 1, 1)$.)

3. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be given by $F(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 - (x^3)^2$. What values does the rank of DF take at different points of $\mathbb{R}^3$? Sketch $F^{-1}(a)$ for $a = -1, 0, 1$. Which of these surfaces (in $\mathbb{R}^3$) is smooth at every point? Give a brief (sentence or two) explanation why. (This question should help give a bit of insight into the general Implicit Function Theorem.)

4. $O(n)$ = "the orthogonal group" = all choices of orthonormal basis $\{\vec{e_1}, \ldots, \vec{e_n}\}$ of $\mathbb{R}^n$ = all $n \times n$ real matrices A such that $AA^T = I$. Here

$$A = \begin{pmatrix} \vec{e_1} \\ \vdots \\ \vec{e_n} \end{pmatrix}$$

(a) $O(4)$ is the solution set of ?? equations in the variables

$$(x^1, x^2, x^3, x^4, y^1, y^2, y^3, y^4, z^1, z^2, z^3, z^4, w^1, w^2, w^3, w^4).$$

Write down these equations. (Note: you can also write these variables as $(\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4})$ where $\vec{e_i} \in \mathbf{R^4}$, which make make this a little easier.)

(b) Write down the ??$\times$16 Jacobian matrix Dg:

(c) Show that $Dg(A)$ has full rank for every $A \in O(4)$. So, by the general Implicit Function Theorem, $O(4)$ is smooth submanifold of $\mathbb{R}^{16}$. (There is a really easy way to do the computation, and a not-too-hard way.)

(d) By the general Implicit Function Theorem, any $A \in O(4)$ is in an open set and some of the variables $(x^1, \ldots, w^4)$ may be written as a differentiable function of the other variables. Give such a choice for $A = I$.