Math 225 Homework 6

Solution

1. *Partial derivatives of differentiable functions need not be continuous.*

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

   Show that f is differentiable at 0, but f' is not continuous at 0.
   Answer When $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. Now when $x = 0$

   $$\begin{aligned} \lim_{h \to 0} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} \\ &= \lim_{h \to 0} h \sin \frac{1}{h} \\ &= 0 \end{aligned}$$

   Hence

   $$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   If f' is continuous at 0, then $\lim_{x \to 0} f'(x) = f'(0) = 0$. But as $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist, then $\lim_{x \to 0} 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ does not exist. Hence f' is not continuous at 0.

   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

   $$f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

   Show that f is differentiable at $(0, 0)$ but that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not continuous at $(0, 0)$.

Answer: Note that for $(x, y) \neq (0, 0)$, $||\sin \frac{1}{\sqrt{x^2+y^2}}|| \leq 1$, hence

$$||(x^2 + y^2) \sin \frac{1}{\sqrt{x^2 + y^2}}|| \leq ||x^2 + y^2|| = ||(x, y)||^2.$$

Thus for any (x, y), $||f(x, y)|| \leq ||(x, y)||^2$. We proved in HW 4 problem 4 that such functions are differentiable at 0. Let's review: $f(0, 0) = 0$ and close to the origin the function is pretty flat, so it is reasonable to guess that $\lambda(h, k) = 0$.

$$\begin{aligned} Df(0,0) &= \lim_{(h,k)\to(0,0)} \frac{||f(0+h, 0+k) - f(0,0) - \lambda(h,k)||}{||(h,k)||} \\ &= \lim_{(h,k)\to(0,0)} \frac{||f(h,k)||}{||(h,k)||} \\ &\leq \lim_{(h,k)\to(0,0)} \frac{||(h,k)||^2}{||(h,k)||} = 0 \end{aligned}$$

Now for $x \neq 0$, $\frac{\partial f}{\partial x} = 2x \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}$. For $x = 0$, we need to use the definition of partial derivatives.

$$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{\sqrt{h^2+0}} - 0}{h} = \lim_{h\to 0} h \sin \frac{1}{h} = 0.$$

Altogether we have

$$\frac{\partial f}{\partial x} = \begin{cases} 2x \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

If $\frac{\partial f}{\partial x}$ is continuous at 0, then $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}|_{(0,0)} = 0$. But

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{(x,y)\to(0,0)} 2x \sin \frac{1}{\sqrt{x^2 + y^2}} - \frac{x}{\sqrt{x^2 + y^2}} \cos \frac{1}{\sqrt{x^2 + y^2}}$$

When $x = y$, $\frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{2x^2}}$ and $\lim_{x\to 0} \frac{1}{\sqrt{2}} \cos \frac{1}{2x^2}$ does not exist. Hence $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}$ does not exist and $\frac{\partial f}{\partial x}$ is not continuous at $(0, 0)$.
As $f(x, y)$ is symmetric in x and y a similar argument shows that $\frac{\partial f}{\partial y}$ exists at $(0, 0)$ but is not differentiable at $(0, 0)$.

2. *The existence of partial derivatives does not imply differentiability.*

   Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

(a) Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere on $\mathbb{R}^2$.

Answer: When $(x, y) \neq (0, 0)$, then $\frac{\partial f}{\partial x} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$. When $(x, y) = (0, 0)$, we need to use the definition, so

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{h.0}{h^2 + 0} \\ &= \lim_{h \to 0} 0 = 0\end{aligned}$$

Altogether we have

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{y(y^2-x^2)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

A similar argument (omitted) shows that

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{x(x^2-y^2)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

thus both partial derivatives exist everywhere.

(b) At what points $(x, y) \in \mathbb{R}^2$ is f differentiable? *You should figure out why your answer to (b) does not contradict Theorem 2.8 on page 31 of Spivak.*

Answer: Away from $(0, 0)$, f is a differentiable function as it is a quotient of two differentiable functions with the denominator not 0. At $(0, 0)$, f is not continuous, so it is not differentiable. (Recall that we proved in the HW that if a function is differentiable at a point, then it is continuous at that point.) To see that f is not continuous at $(0, 0)$, consider what happens when $x = 0$, then $\lim_{(0,y)\to(0,0)} f(0, y) = 0$. However when $x = y$, $\lim_{(0,y)\to(0,0)} f(0, y) = 1/2 \neq 0$. This answer does not contradict Theorem 2.8 as the partial derivatives are not continuous at $(0, 0)$ and this was a requirement of the theorem. To see this consider $\frac{\partial f}{\partial x}$.

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{(x,y)\to(0,0)} \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

When $y = 0$, this becomes $\lim_{(x,y)\to(0,0)} 0 = 0$. When $y = x^2$ this becomes $\lim_{(x,y)\to(0,0)} \frac{x^2(x^4-x^2)}{(x^2+x^4)^2} = \lim_{(x,y)\to(0,0)} \frac{x^4(x^2-1)}{x^4(1+x^2)^2} = -1 \neq 0$. Hence $\frac{\partial f}{\partial x}$ is not continuous at $(0, 0)$. A similar argument (omitted) holds for $\frac{\partial f}{\partial y}$.

3. *Jacobian Matrix*

Find the Jacobian Matrix for the following functions.

(a) $s : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$s(\rho, \theta, \phi) = (x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

(b) $g : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$g(x, y, z) = (u, v, w) = (x \cos(2\pi y), x \sin(2\pi y), z).$$

Everyone got these right.

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Show that f is differentiable at 0 and that $f'(0) = \frac{1}{2}$.
Answer: Using the definition of derivative we find

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h}{2} + h^2 \sin(\frac{1}{h})}{h} \\ &= \lim_{h\to 0} \frac{1}{2} + h \sin(\frac{1}{h}) \\ &= \frac{1}{2} \end{aligned}$$

(b) Show that there is no open interval containing 0 on which f is one-to-one.

Answer: There are a couple of ways to do this, here is one. If there is an interval $(-a, a)$ where f is one-to-one then f is either increasing or decreasing. That is if f is one-to-one then $f' \geq 0$ or $f' \leq 0$. Now $f' = \frac{1}{2} + 2x \sin(\frac{1}{x})) - \cos(\frac{1}{x})$. If $x = \frac{1}{2n\pi}$ for $n = \pm 1, \pm 2, \pm 3, \ldots$ then $\sin(\frac{1}{x}) = \sin(2n\pi) = 0$ and $\cos(\frac{1}{x}) = \cos(2n\pi) = 1$so $f'(x) = \frac{1}{2} - 1 = -\frac{1}{2}$. On the other hand if $x = \frac{1}{2(n+1)\pi}$ for $n = \pm 1, \pm 2, \pm 3, \ldots$ $\sin(\frac{1}{x}) = \sin(2(n+1)\pi) = 0$ and $\cos(\frac{1}{x}) = \cos(2(n+1)\pi) = -1$ so $f'(x) = \frac{1}{2} + 1 = \frac{1}{2}$. Hence f can't be one to one on any interval $(-a, a)$.

Why doesn't this contradict the Inverse Function Theorem?

The Inverse function theorem requires the function to be continuously differentiable at the point in question in order to conclude there is a neighborhood about that point where the function is one to one. We have found that

$$f'(x) = \begin{cases} \frac{1}{2} + 2x \sin(\frac{1}{x})) - \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

However $\lim_{x\to 0} f'(x)$ does not exist (as $\lim_{x\to 0}\cos(\frac{1}{x})$ does not exist). Hence $f'(x)$ is not continuous at 0.

5. *Chain rule.*

   (a) Problem 2-28 part (b) on page 33 of Spivak.

   Answer: Here $F : \mathbb{R}^3 \to \mathbb{R}$ with $F(x, y, z) = f(g(x+y), h(y+z))$. Splitting this up, we find $f(u, v)$ with $f : \mathbb{R}^2 \to \mathbb{R}$, $g, h : \mathbb{R} \to \mathbb{R}$ with $u = g(s)$ and $v = h(t)$ say, $s, t : \mathbb{R}^3 \to \mathbb{R}$ with $s(x, y, z) = x + y$ and $t(x, y, z) = y + z$.

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}\frac{\partial t}{\partial x} \\ &= \frac{\partial f}{\partial u} g'(x+y)(1) + \frac{\partial f}{\partial u} h'(y+z)(0) = \frac{\partial f}{\partial u} g'(x+y).\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}\frac{\partial t}{\partial y} \\ &= \frac{\partial f}{\partial u} g'(x+y)(1) + \frac{\partial f}{\partial u} h'(y+z)(1) = \frac{\partial f}{\partial u} g'(x+y) + \frac{\partial f}{\partial u} h'(y+z)\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}\frac{\partial t}{\partial z} \\ &= \frac{\partial f}{\partial u} g'(x+y)(0) + \frac{\partial f}{\partial u} h'(y+z)(1) = \frac{\partial f}{\partial u} h'(y+z)\end{aligned}$$

   (b) The temperature at a point (x, y) in the plane is given by $T(x, y) = x^2y + 3xy^4$. An ant crawls on the plan such that its position after t seconds is given by $x = \sin 2t$ and $y = \cos t$. Find the rate of change of temperature along the ant's path when $t = 0$.

   Answer: $T : \mathbb{R}^2 \to \mathbb{R}$ with $T(x, y)$ and $x : \mathbb{R} \to \mathbb{R}$ with $x(t) = \sin 2t$ and $y : \mathbb{R} \to \mathbb{R}$ with $y(t) = \cos t$. Hence

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial t} \\ &= (2xy + 3y^4)2\cos(2t) + (x^2 + 12xy^3)(-\sin t)\end{aligned}$$

   When $t = 0$, $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Hence $\frac{\partial T}{\partial t} = (0 + 3)2 = 6$ degrees per second.