Math 225 Homework 6

Due 4pm on Fri March 6, 2009.

1. *Partial derivatives of differentiable functions need not be continuous.*

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

   Show that f is differentiable at 0, but f' is not continuous at 0.

   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

   $$f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

   Show that f is differentiable at $(0, 0)$ but that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not continuous at $(0, 0)$.

2. *The existence of partial derivatives does not imply differentiability.*

   Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$:

   $$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

   (a) Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere on $\mathbb{R}^2$.

   (b) At what points $(x, y) \in \mathbb{R}^2$ is f differentiable? *You should figure out why your answer to (b) does not contradict Theorem 2.8 on page 31 of Spivak.*

3. *Jacobian Matrix*

   Find the Jacobian Matrix for the following functions.

(a) $s : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$s(\rho, \theta, \phi) = (x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

(b) $g : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$g(x, y, z) = (u, v, w) = (x \cos(2\pi y), x \sin(2\pi y), z).$$

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Show that f is differentiable at 0 and that $f'(0) = \frac{1}{2}$.

(b) Show that there is no open interval containing 0 on which f is one-to-one.

Why doesn't this contradict the Inverse Function Theorem?

5. *Chain rule.*

(a) Problem 2-28 part (b) on page 33 of Spivak.

(b) The temperature at a point (x, y) in the plane is given by $T(x, y) = x^2 y + 3xy^4$. An ant crawls on the plan such that its position after t seconds is given by $x = \sin 2t$ and $y = \cos t$. Find the rate of change of temperature along the ant's path when $t = 0$.