Math 225 Homework 5

Solutions

1. Problem 2.10 parts b, h, page 22 Spivak. I'd like you to do this question only using information about the derivative you have up to page 22 of the text. You'll find it helpful to look at the example done on page 22, as well as the example I did in class.

   Answer (b) $f : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $f(x, y, z) = (x^y, z)$. Here $f^1(x, y, z) = x^y = e^{(\ln x)y}$ and $f^2(x, y, z) = z$. We know $Df(x, y, z) = (Df^1(x, y, z), Df^2(x, y, z))$, where $Df^i(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$.

   Now $f^2(x, y, z) = \pi^3(x, y, z) = z$ and we proved in HW 4 that $D\pi^3(x, y, z) = \pi^3$ and has matrix $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Note that $f^1(x, y, z) = [e \circ (\pi^2 \ln(\pi^1))](x, y, z)$. Thus

$$\begin{aligned} Df^1(x, y, z) &= De^{y \ln x} D(\pi^2 \ln(\pi^1))(x, y, z) \\ &= e^{y \ln x}[D\pi^2(x, y, z) \ln x + yD \ln(\pi^1)(x, y, z)] \\ &= e^{y \ln x}[\pi^2 + y\frac{1}{x} D\pi^1(x, y, z)] \\ &= e^{y \ln x}[\pi^2 + \frac{y}{x}\pi^1] \end{aligned}$$

   In matrix form

$$Df^1(x, y, z) = x^y \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \frac{y}{x} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} x^y \frac{y}{x} & x^y & 0 \end{bmatrix} = \begin{bmatrix} yx^{y-1} & x^y & 0 \end{bmatrix}$$

   Answer (h) $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by $f(x, y, z) = \sin(xy) = [\sin \circ(\pi^1\pi^2)](x, y, z)$, thus

$$\begin{aligned} Df(x, y, z) &= D \sin(xy) Dp(\pi^1\pi^2)(x, y, z) \\ &= \cos(xy)[yD\pi^1(x, y, z) + xD\pi^2(x, y, z)] \\ &= \cos(xy)[y\pi^1 + x\pi^2] \end{aligned}$$

   In matrix form

$$Df(x, y, z) = \cos(xy) \left(y \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + x \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} y \cos(xy) & x \cos(xy) & 0 \end{bmatrix}$$

2. Problem 2.13 page 22 Spivak.

Answer (a) Let $x, y \in \mathbb{R}^n$, then $IP : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$IP(x,y) = \langle x, y \rangle = x^1y^1 + x^2y^2 + \dots x^ny^n = [\pi^1\pi^{n+1} + \pi^2\pi^{n+2} \dots + \pi^n\pi^{2n}](x,y)$$

Now use $D(f+g)(a) = Df(a) + Dg(a)$ and the product rule to conclude

$$\begin{aligned} DIP(x,y) &= D(\pi^1\pi^{n+1})(x,y) + D(\pi^2\pi^{n+2})(x,y) \dots + D(\pi^n\pi^{2n})(x,y) \\ &= \pi^1(x,y)D\pi^{n+1}(x,y) + \pi^{n+1}(x,y)D\pi^1(x,y) + \pi^2(x,y)D\pi^{n+2}(x,y) \\ &+ \pi^{n+2}(x,y)D\pi^2(x,y) + \dots + \pi^n(x,y)D\pi^{2n}(x,y) + \pi^{2n}(x,y)D\pi^n(x,y) \\ &= x^1\pi^{n+1} + y^1\pi^1 + x^2\pi^{n+2} + y^2\pi^2 + \dots + x^n\pi^{2n} + y^n\pi^n \end{aligned}$$

In matrix form $\pi^i = \begin{bmatrix} 0 & \dots & 1 & 0\dots & 0 \end{bmatrix}$, Hence

$$DIP(x,y) = \begin{bmatrix} x^1 & x^2 & \dots & x^n & y^{n+1} & \dots & y^{2n} \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

Answer (b) Assume $f, g : \mathbb{R} \to \mathbb{R}^n$ are differentiable and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = \langle f(t), g(t) \rangle = IP(f(t), g(t))$. Then by the chain rule

$$\begin{aligned} Dh(a) &= DIP(f(a), g(a))D(f,g)(a) \\ &= \begin{bmatrix} g(a) & f(a) \end{bmatrix} \begin{bmatrix} Df(a) \\ Dg(a) \end{bmatrix} \\ &= \langle g(a), Df(a)^T \rangle + \langle f(a), Dg(a)^T \rangle \end{aligned}$$

Note that $\begin{bmatrix} g(a) & f(a) \end{bmatrix}$ is a $1 \times 2n$ matrix and $\begin{bmatrix} Df(a) \\ Dg(a) \end{bmatrix}$ is a $2n \times 1$ matrix. Thus the matrix product is equivalent to taking the inner product of vectors $g(a)$ and $Df(a)^T$ and adding this to the inner product of vectors $f(a)$ and $Dg(a)^T$.

Answer (c) Assume $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable and $||f(t)|| = 1$ for all t. Rewrite this as $1 = ||f(t)|^2| = \langle f(t), f(t) \rangle$. Differentiate both sides with respect to t gives

$$0 = \langle f(t), Df(t)^T \rangle + \langle f(t), Df(t)^T \rangle = 2\langle f(t), Df(t)^T \rangle.$$

(Note we are using the fact that $\langle a, b \rangle = \langle b, a \rangle$.)

Answer (d) $f(t) = t$ works. Because $|f|(t) = |t|$, which is not differentiable at 0.

3. Problem 2.16 page 25 Spivak.

Answer: We are given $f, f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, a differentiable function and its inverse, also differentiable. Observe that $f \circ f^{-1}(x) = x$, so that $f \circ f^{-1}$ is the identity map I.

The identity map is a linear map. This means that $DI(a) = I$ for any point $a \in \mathbb{R}^n$. Given $a \in \mathbb{R}^n$, let $b = f^{-1}(a)$. The chain rule gives

$$D(f \circ f^{-1})(a) = Df(f^{-1}(a))Df^{-1}(a) = Df(b)Df^{-1}(a) = I.$$

In matrix form $I = I_n$ the $n \times n$ identity matrix. Thus $Df^{-1}(a) = [Df(b)]^{-1}$, where we mean the matrix inverse of $Df(b)$.

4. Problem 2.17 parts e, g, i page 28 Spivak

   Answer (e) $f(x,y,z) = x^{y^z} = e^{\ln x (y^z)} = e^{(\ln x)e^{z \ln y}}$. Hence $\frac{\partial f}{\partial x} = y^z x^{y^z - 1}$, $\frac{\partial f}{\partial y} = zy^{z-1} \ln x e^{\ln x (y^z)} = zy^{z-1}(\ln x)x^{y^z}$, $\frac{\partial f}{\partial z} = x^{y^z}(\ln x)(\ln y)e^{z \ln y}$.

   Answer (g) $f(x,y,z) = (x+y)^z = e^{z \ln(x+y)}$. Hence $\frac{\partial f}{\partial x} = z(x+y)^{z-1}$, $\frac{\partial f}{\partial y} = z(x+y)^{z-1}$, $\frac{\partial f}{\partial z} = \ln(x+y)(x+y)^z$.

   Answer (i) $f(x,y) = [\sin(xy)]^{\cos 3}$, hence $\frac{\partial f}{\partial x} = \cos 3[\sin(xy)]^{\cos 3 - 1} \cos(xy)y$, $\frac{\partial f}{\partial y} = \cos 3[\sin(xy)]^{\cos 3 - 1} \cos(xy)x$

5. Problem 2.20 page 28 Spivak

   Answer (a) $f(x,y) = g(x)h(y)$, then $\frac{\partial f}{\partial x} = g'(x)h(y)$, $\frac{\partial f}{\partial y} = g(x)h'(y)$.

   Answer (b) $f(x,y) = g(x)^{h(y)} = e^{h(y) \ln g(x)}$ $\frac{\partial f}{\partial x} = g'(x)h(y)g(x)^{h(y)-1}$, $\frac{\partial f}{\partial y} = \ln h(y) g(x)^{h(y)}$.

   Answer (c) $f(x,y) = g(x)$ $\frac{\partial f}{\partial x} = g'(x)$, $\frac{\partial f}{\partial y} = 0$.

   Answer (d) $f(x,y) = g(y)$ $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = g'(y)$.

   Answer (e) $f(x,y) = g(x+y)$ $\frac{\partial f}{\partial x} = g'(x+y)$, $\frac{\partial f}{\partial y} = g'(x+y)$.

6. Problem 2.24 parts page 28 Spivak

   This question was not quite done correctly, even though I gave most of you full points. Please read through the solution.

   Answer: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

   $$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   If $x \neq 0$, then $\frac{\partial f}{\partial y} = x\frac{x^2-y^2}{x^2+y^2} + xy(\frac{x^2-y^2}{x^2+y^2})'$, which we evaluate at $(x,0)$, giving $\frac{\partial f}{\partial y} = x\frac{x^2-0}{x^2+0} + 0x(\frac{x^2-y^2}{x^2+y^2})' = x$. If $x = 0$, then we use the definition of partial derivative.

   $$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

Similarly

$$\frac{\partial f}{\partial x} = \begin{cases} -y & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1 \qquad \frac{\partial^2 f}{\partial y \partial x} = -1$$