Math 225 Homework 4

Solutions

1. Recall the projection maps $\pi^i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ defined by

   $$\pi^i(x_1, x_2, \ldots, x_n) = x_i.$$

   Use the definition of differentiability (on p. 16) to show that π^i is differentiable. (Hint: take a guess at what the linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}$ is, then show it satisfies the definition.) What is the matrix that represents $\lambda(h)$?

   Answer: We expect that $\lambda(h) = \pi^i(h)$. Given any point $x \in \mathbb{R}^n$ we compute the following limit.

   $$\lim_{h\to 0} \frac{|||\pi^i(x+h) - \pi^i(x) - \lambda(h)||}{||h||} = \lim_{h\to 0} \frac{|||(x_i + h_i) - x_i - h_i)||}{||h||} = \lim_{h\to 0} \frac{0}{||h||} = 0$$

   Alternatively you could note that the map π^i is linear and show the numerator is zero by noting that $\pi^i(x+h) = \pi^i(x) + \pi^i(h)$. In either case we have proven that at any point $x \in \mathbb{R}^n$ π^i is differentiable and $D\pi^i(x) : \mathbb{R}^n \to \mathbb{R}$ is defined by $D\pi^i(x)(h) = \pi^i(h) = h^i$. $D\pi^i(x)$ is represented by the $1 \times n$ matrix $\begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}$ where the 1 is in the ith position.

2. Problem 2.1 page 17 Spivak. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ then it is continuous at a,

   Answer: We know there is a linear function $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

   $$\lim_{h\to 0} \frac{|||f(a+h) - f(a) - \lambda(h)||}{||h||} = 0.$$

   We are trying to prove that f is continuous at a, that is

   $$\lim_{x\to a} f(x) = f(a).$$

   First note that $\lim_{x\to a} f(x) = f(a)$ is equivalent to $\lim_{h\to 0} f(a+h) = f(a)$, which is equivalent to $\lim_{h\to 0} f(a+h) - f(a) = 0$ which is equivalent to $\lim_{h\to 0} ||f(a+h) - f(a)|| = 0$ (because norm is continuous). To summarize, to prove f continuous at a it is sufficient to prove that $\lim_{h\to 0} ||f(a+h) - f(a)|| = 0$. Now,

$$\begin{aligned}0 &\leq ||f(a+h) - f(a)|| \\ &\leq ||f(a+h) - f(a) - \lambda(h) + \lambda(h)|| \\ &\leq ||f(a+h) - f(a) - \lambda(h)|| + ||\lambda(h)|| \\ &\leq \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||}||h|| + M||h||,\end{aligned}$$

where $M > 0$ is a positive constant (from problem 1-10). If we take the limit at $h \to 0$, then because of the assumption of differentiability, we get

$$0 \leq \lim_{h\to 0} ||f(a+h) - f(a)|| \leq 0,$$

which was what we wanted.

You can also prove this result by $\epsilon - \delta$ arguments as follows. First note, by the argument above that

$$0 \leq ||f(a+h) - f(a)|| \leq \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||}||h|| + M||h||.$$

Fix $\epsilon > 0$. As f is differentiable at a, we know there is a $\delta_1 > 0$ such that $0 < ||h|| < \delta_1$ means that $\frac{||f(a+h)-f(a)-\lambda(h)||}{||h||} < \epsilon/2$. As $\lambda(h)$ is differentiable at a, there is a δ_2 such that $0 < ||h|| < \delta_2$ means that $||\lambda(h)|| \leq M||h|| < \epsilon/2$.

Choose $\delta = \min(\delta_1, \delta_2, 1)$. Then $0 < ||h|| < \delta$ means

$$\begin{aligned}||f(a+h) - f(a)|| &\leq \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||}||h|| + M||h|| \\ &< (\epsilon/2)\delta + \epsilon/2 \\ &< \epsilon.\end{aligned}$$

3. Problem 2.3 page 17 Spivak. (Hint: modify the arguments we gave in class for Problem 2.2.)

   Answer: Functions $f : \mathbb{R}^2 \to \mathbb{R}$ which are independent of the first variable are functions if for each $y \in \mathbb{R}$, $f(x_1, y) = f(x_2, y)$ for any $x_1, x_2 \in \mathbb{R}^2$.

   Claim 1: f is independent of the first variable if and only if there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(y) = f(x, y)$.

   Proof. Assume f is independent of the first variable and define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = f(0, y)$. Then $g(y) = f(x, y) = f(0, y)$ by assumption. Let g be the function with $g(y) = f(x, y)$, then f is independent of first variable since $f(x_1, y) = g(y) = f(x_2, y)$. □

Claim 2: For such a function, $f'(a,b) = g'(b)$.

Proof.

$$\begin{aligned}\lim_{(h,k)\to(0,0)} \frac{||f(a+h,b+k) - f(a,b) - g'(b)||}{||(h,k)||} &= \lim_{(h,k)\to(0,0)} \frac{||g(b+k) - g(b) - g'(b)||}{||(h,k)||} \\ &\leq \lim_{k\to 0} \frac{||g(b+k) - g(b) - g'(b)||}{||k||} \\ &= 0\end{aligned}$$

□

Note that the inequality is because $\sqrt{k^2} \leq \sqrt{h^2+k^2}$ so $\frac{1}{\sqrt{k^2}} \geq \frac{1}{\sqrt{h^2+k^2}}$. The last equality is the definition of $g'(b)$.

Problem 2.7 page 18 Spivak.

Answer: We assume that $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that $||f(x)|| \leq ||x||^2$. This means that $0 \leq ||f(0)|| \leq ||0|| = 0$ so $f(0,0) = 0$. We want to show that f is differentiable at 0. Using the assumption, we see that $\lambda : \mathbb{R}^2 \to \mathbb{R}$ defined by $\lambda(h) = 0$ makes sense.

$$\begin{aligned}\lim_{(h,k)\to(0,0)} \frac{||f(h,k) - f(0,0) - \lambda(0,0)||}{||(h,k)||} &= \lim_{(h,k)\to(0,0)} \frac{||f(h,k) - 0 - 0||}{||(h,k)||} \\ &= \lim_{(h,k)\to(0,0)} \frac{||f(h,k)||}{||(h,k)||} \\ &\leq \lim_{(h,k)\to(0,0)} \frac{||(h,k)||^2}{||(h,k)||} \\ &= \lim_{(h,k)\to(0,0)} ||(h,k)|| = 0.\end{aligned}$$