Math 225 Homework 3

Solutions

1 Problems

1. Give an example of a function $\mathbb{R}^2 \to \mathbb{R}$ which is not continuous at a point (a, b) in its domain. (You must prove that it is not continuous at the point in question.)

   Answer: Reasonably well done. Let me do one example to illustrate a good solution. Given $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

   $$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{1}{x^2+y^2} & \text{otherwise.} \end{cases}$$

   Then f is not continuous at $(0, 0)$. To see this first observe that if $||(x, y)|| = \sqrt{x^2 + y^2} < 1$, then $f(x, y) > 1$. Set $\epsilon = 1$. Then for any $\delta > 0$, when $0 \leq ||(x, y) - (0, 0)|| = ||(x, y)|| < \delta$, then $||f(x, y) - f(0, 0)|| < 1$ is not true! Why? $||f(x, y) - f(0, 0)|| = ||f(x, y)|| = \frac{1}{x^2+y^2}$. But, whatever δ is, there is always at least one (x, y) with $||(x, y)|| < 1$ and for this (x, y), $f(x, y) > 1$. (For example if $\delta = 1/2$, then pick $(x, y) = (1/10, 0)$. Here $||(1/10, 0)|| = 1/10 < 1/2$ and $f(1/10, 0) = 100 > 1$.)

2. Problem 1.23 page 13 Spivak.

   Answer: $A \subset \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ such that $f(x_1, \ldots, x_n) = (f^1(x_1, \ldots, x_n), \ldots, f^m(x_1, \ldots, x_n))$. Let $a \in A$.

   Assume that $\lim_{x \to a} f(x) = b$. We want to show that for each $i = 1, 2 \ldots, m$, $\lim_{x \to a} f^i(x) = b^i$. Take $\epsilon > 0$ and note that because of the assumption, there is a $\delta > 0$, such that when $0 < ||x - a|| < \delta$, then $||f(x) - b)|| < \epsilon$. Note that $||f(x) - b|| = \sqrt{(f^1(x) - b^1)^2 + \cdots + (f^m(x) - b^m)^2}$. So when $0 < ||x - a|| < \delta$ then $||f^i(x) - b^i|| = \sqrt{(f^i(x) - b^i)^2} \leq ||f(x) - b|| < \epsilon$. This is exactly what we needed to show $\lim_{x \to a} f^i(x) = b^i$ for each i.

   Now assume that for each $i = 1, 2 \ldots, m$, $\lim_{x \to a} f^i(x) = b^i$. From HW 1 we know that $||f(x) - b|| \leq ||f^1(x) - b^1|| + ||f^2(x) - b^2|| + \cdots + ||f^m - b^m||$. Now take $\epsilon > 0$. As $\lim_{x \to a} f^1(x) = b^1$, we know that for ϵ/m, there is a δ_1 such that when $0 < ||x - a|| < \delta_1$ then $||f^1(x) - b^1|| < \epsilon/m$. This is true for each $f^i(x)$. Take

$\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then when $0 < ||x - a|| < \delta$, then $||f(x) - b|| \leq ||f^1(x) - b^1|| + ||f^2(x) - b^2|| + \cdots + ||f^m - b^m|| < \epsilon/m + \cdots + \epsilon/m = \epsilon$.

3. Problem 1.24 page 13 Spivak. $\lim_{x \to a} f(x) = f(a)$ if and only if $\lim_{x \to a} f^i(x) = f^i(a)$ for each $i = 1, 2 \ldots, m$.

   Answer: This question is an application of the previous problem where $b = f(a)$.

4. Problem 1.10 page 5 Spivak.

   Answer: $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and can be represented by a matrix $A = (a_{ij})$ and $T(x) = Ax = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is an $n \times 1$ matrix. The ith entry is given by $b_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m$, or the ith row of A dotted with x. Let A_i be the ith row of A, and so by the Cauchy-Schwarz inequality, $||b_i|| = ||A \cdot x|| \leq ||A_i|| \, ||x||$. Let $M = \max_i A_i$, then $||T(x_1, \ldots, x_m)|| \leq ||b_1|| + ||b_2|| + \ldots ||b_m|| \leq M||x||$.

5. Problem 1.25 page 13 Spivak. Prove that linear functions are continuous.

   Answer: Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear function and let $a \in \mathbb{R}^m$. We aim to show that $\lim_{x \to a} T(x) = T(a)$.

   Fix $\epsilon > 0$ and set $\delta = M/\epsilon$. When $0 < ||x - a|| < \delta$, then $||T(x) - T(a)|| = ||T(x - a)|| \leq M||x - a|| < \epsilon$.

2 More problems

Everyone did a great job.