Math 225 Homework 2

Due Friday February 6, 2009.

When writing up your solutions, pay attention to what you write. I'm interested in seeing proofs written rigorously. What does this mean? Good proofs are:

- Correct — ideally, every statement should follow from axioms or from what has been proved before.
- Concise — a proof should not contain anything that is not necessary.
- Readable — Human beings both write and read proofs. Don't be afraid to explain in words what you are doing. For example, before embarking on a long computation, it is a good idea to explain what you are doing and why you are doing it.

1 Problems

1. Which of the following functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ are linear transformations? (You must justify your answer by showing why the definition of a linear transformation is true or why it fails.)
   (a) $f(x_1, x_2, x_3) = (2x_2, x_2 + 2, 2x_2)$
   (b) $g(x_1, x_2, x_3) = (2x_2, 3x_3, x_1)$
   (c) $h(x_1, x_2, x_3) = (x_2 - x_3, x_1x_2, x_1 - x_2)$
2. Consider the function $f : \mathbb{R} \to \mathbb{R}^3$ defined by $f(t) = (\cos t, \sin t, \sin 4t)$.
   (a) What is the domain of this function?
   (b) The image of this function is a parametric curve in $\mathbb{R}^3$. Sketch this curve from $t = -\frac{\pi}{4}$ to $t = \frac{\pi}{4}$.
3. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}.$$

   (a) You are given the unit square $S = [0, 1] \times [0, 1] \in \mathbb{R}^2$. Where does this square get mapped to under T? Please draw a picture of $T(S)$.

(b) What is the area of $T(S)$?

(c) What happens to the orientation of the boundary of S under the transformation T? (Hint: Imagine an ant walking around the boundary of S counter-clockwise starting at the origin. Does the ant walk around the boundary of $T(S)$ counter-clockwise?)

(d) What is the determinant of the matrix A? What do you notice? (We'll explore this connection in later homework assignments.)

4. Let $A = [0, \infty) \times \mathbb{R} \times \mathbb{R} \subset \mathbb{R}^3$ and let the function $g : A \to \mathbb{R}^3$ be defined by

$$g(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

(a) Draw a picture of $\phi = \frac{3\pi}{4}$ and $\theta = \frac{\pi}{3}$

(b) On a separate picture draw the image under g of the cube $[0, 2] \times [\frac{\pi}{3}, \frac{\pi}{2}] \times [\frac{\pi}{2}, \frac{3\pi}{4}]$.

5. Problem 1.14 page 10 Spivak.

6. Problem 1.16 page 10 Spivak.

7. In a real analysis course you will see an open set defined in $\mathbb{R}^n$ as an open ball. More formally, given point $a \in \mathbb{R}^n$, and a constant $r > 0$ the open ball is denoted $B_r(a)$ and is the the set of all points that are within distance r of the point a, namely

$$B_r(a) := \{x \in \mathbb{R}^n \mid ||x - a|| < r\}.$$

(a) Draw a picture of $B_1(2)$. (This is a subset of $\mathbb{R}$.)

(b) Draw a picture of $B_{\frac{1}{2}}(a)$, where $a = (0, 0, \frac{1}{2}) \in \mathbb{R}^3$.

8. Prove that $B_r(a)$ is an open set in $\mathbb{R}^n$. (Hint: you must show that the definition of an open set given in Spivak is true.)

9. Consider an open rectangle $R = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$. Show that for each $x \in R$, there is an open ball B such that $x \in B \subset R$.

   Remark: Questions 8 and 9 together show that the definition of an open set given by rectangles (in Spivak) and the definition of an open set given using balls (in a typical real analysis course) are equivalent.

10. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x + 1$, prove that f is continuous at $x = 1$. Use the $\epsilon - \delta$ definition of continuity to do this.

11. If $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{if } x \neq 1 \\ 0 & \text{if } x = 0 \end{cases},$$

then prove (use the $\epsilon - \delta$ definition) that f is not continuous at $x = 1$.

12. If the two functions $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are both continuous at $x = a$, prove that the function $f + g : \mathbb{R}^n \to \mathbb{R}^m$ is also continuous at $x = a$.

13. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a and $g : \mathbb{R}^m \to \mathbb{R}^p$ is continuous at $f(a)$, prove that the composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is continuous at a.

2 Reminder — find my office and have a brief chat

This task is worth one quarter of your grade for HW 1. Remember that you have until February 13 to complete it.

This means you'll know where my office is when you have questions. I'll touch base with all of you, learning the names of students I haven't already met and catching up with those of you I've taught previously.