Math 225 Homework 2

Brief Solutions

1 Problems

1. Which of the following functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ are linear transformations?

   Nicely done everyone.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}^3$ defined by $f(t) = (\cos t, \sin t, \sin 4t)$.

   Nice job.

3. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}.$$

   a. You are given the unit square $S = [0,1] \times [0,1] \in \mathbb{R}^2$. Where does this square get mapped to under T? Please draw a picture of $T(S)$.

   Good job.

   b. What is the area of $T(S)$?

   You find it by taking the area of the large rectangle that contains the parallelogram, then subtracting the area of the 4 triangles outside the parallelogram. Namely $7 = 15 - 0.5(3 \times 1 + 3 \times 1 + 5 \times 1 + 5 \times 1)$.

4. Let $A = [0, \infty) \times \mathbb{R} \times \mathbb{R} \subset \mathbb{R}^3$ and let the function $g : A \to \mathbb{R}^3$ be defined by

$$g(\rho, \theta, \phi) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi).$$

   Good job.

5. Problem 1.14 page 10 Spivak.

   Answer: Let U_i (i=1,2,...) be a collection of open sets and let $U := \bigcup_i U_i$. Given any $x \in U$ we need to show there is an open rectangle R, such that $x \in R \subset U$. But by definition of union, $x \in U_i$ for some i and by the definition of U_i open, there is an R_i with $x \in \mathbb{R}_i \subset U_i$. As $U_i \subset U$ we are done.

Consider two open sets U and V. If $U \cap V = \emptyset$, then the intersection is open, since $\emptyset$ is open. Assume $U \cap V \neq \emptyset$ and take $x \in U \cap V$. By definition of intersection $x \in U$ and $x \in V$ and by definition of open sets, there are open rectangles S and T such that $x \in S \subset U$ and $x \in T \subset V$. Then $S \cap T$ is an open rectangle in both U and V, that is $x \in S \cap T \subset U \cap V$.

This proof carries over when proving that the intersection of a finite number of open sets is open as a finite intersection of rectangles will be an open rectangle. If there are an infinite number of sets, it may be possible for the intersection of open rectangles to be a point, which is closed. For example, consider the collection of open sets in $\mathbb{R}$: $(1 - \frac{1}{n}, 1 + \frac{1}{n})$ for $n = 1, 2, 3, \ldots$. The intersection of these sets is the closed set $\{1\}$.

6. Problem 1.16 page 10 Spivak.

   Parts (a) and (b) were well done. For part (c) we are looking at the set $A = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \text{each } x_i \text{ is rational}\}$. To find the interior, we take all points in $a \in A$, such that there is an open rectangle about a is contained in A. However an irrational number may be found arbitrarily close to any ration number, so no such rectangle can be found, so the interior of A is empty. In a similar way, the exterior of A is empty. Take a point $b \in \mathbb{R}^n \setminus A$ (at least one component b_i is irrational. Then as a rational number can be found arbitrarily close to any irrational number, we can not find an open rectangle lying in $\mathbb{R}^n \setminus A$. The boundary of A consists of those points in $\mathbb{R}^n$ where any open rectangle contains points in both A and $\mathbb{R}^n \setminus A$. From the previous arguments we see that all points of $\mathbb{R}^n$ satisfy this condition.

7. (a) Draw a picture of $B_1(2)$. (This is a subset of $\mathbb{R}$.)

   (b) Draw a picture of $B_{\frac{1}{2}}(a)$, where $a = (0, 0, \frac{1}{2}) \in \mathbb{R}^3$.

   Great job.

8. Prove that $B_r(a)$ is an open set in $\mathbb{R}^n$. (Hint: you must show that the definition of an open set given in Spivak is true.)

   Answer: Given an $x \in B_r(a)$, we must show there is an open rectangle R such that $x \in R \subset B_r(a)$. So the best answers showed (a) how to construct the rectangle and (b) the rectangle lies in $B_r(a)$.

   Let $t = ||x - a||$ and let $d = r - t$ (that is, d is the distance from x to the boundary of R). Let x be the center of an open rectangle defined by $R = (x_1 - p, x_1 + p) \times \cdots \times (x_n - p, x_n + p)$. We require this rectangle to be inside $B_r(a)$, hence the half-diagonal from x to one corner $(x_1 + p, x_2 + p, \ldots, x_n + p)$ must lie in $B_r(a)$. The length of that half-diagonal must be less than d, or $\sqrt{np^2} = p\sqrt{n} < d$, or $p < d/\sqrt{n}$. We can always find such a p, hence $B_r(a)$ is open.

9. Consider an open rectangle $R = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$. Show that for each $x \in R$, there is an open ball B such that $x \in B \subset R$.

Answer: Given an $x \in R$, we meed to construct a open ball B centered at x, then show it lies in R. For x in R, take the minimum of the distance to all of the boundary edges. That is let r be the minimum of $|x_1 - a_1|, |x_1 - b_1|, |x_2 - a_2|, |x_2 - b_2|, \ldots, |x_n - a_n|, |x_n - b_n|$. Then $B_r(x)$ is a subset of $(x_1 - r, x_1 + r) \times \cdots \times (x_n - r, x_n + r) \subset R$.

10. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x + 1$, prove that f is continuous at $x = 1$. Use the $\epsilon - \delta$ definition of continuity to do this.

    This was very well done - good job everyone!

    In fact it is easy to show that f is continuous at any $x = a$. Pick some constant $\epsilon > 0$, then consider $|f(x) - f(a)| = |3x + 1 - 3a - 1| = |3(x - a)| = 3|x - a|$. If $\delta < \epsilon/3$, then when $0 < |x - a| < \epsilon/3$, then $|f(x) - f(a)| = 3|x - a| < \epsilon$.

11. If $f : \mathbb{R} \to \mathbb{R}$ is defined by

    $$f(x) = \begin{cases} \frac{1}{1-x} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases},$$

    then prove (use the $\epsilon - \delta$ definition) that f is not continuous at $x = 1$.

    Take $\epsilon = 1$ and note that if $|1 - x| < 1$ then $|\frac{1}{1-x}| > 1$. Take any constant $\delta > 0$. Now consider $|f(x) - f(1)| = |\frac{1}{1-x} - 0| = |\frac{1}{1-x}|$. It is not true that $|f(x) - f(1)| < 1$ for each x in $0 < |x - 1| < \delta$, as there is always some value of x for which $|1 - x| < 1$.

12. If the two functions $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are both continuous at $x = a$, prove that the function $f + g : \mathbb{R}^n \to \mathbb{R}^m$ is also continuous at $x = a$.

    Answer: Given a constant $\epsilon > 0$, we need to construct a constant $\delta > 0$, such that when $0 < ||x - a|| < \delta$, then $||(f + g)(x) - (f + g)(a)|| < \epsilon$. Notice in question 10, that you took the ϵ, messed with the function $3x + 1$ and from there constructed the δ. We are going to do something similar in this question. However, we don't have a specific function to play with. Instead we have the knowledge that the functions are continuous. This will provide us with the tools to figure out what δ is.

    Let's have a play: $||(f + g)(x) - (f + g)(a)|| = ||f(x) + g(x) - (f(a) + g(a))|| = ||f(x) - f(a) + g(x) - g(a)|| \leq ||f(x) - f(a)|| + ||g(x) - g(a)||$. So if we get $||f(x) - f(a)|| + ||g(x) - g(a)|| < \epsilon$, then $||(f + g)(x) - (f + g)(a)|| < \epsilon$ too.

    We just need to figure out when $||f(x) - f(a)|| < \epsilon/2$ and $||g(x) - g(a)|| < \epsilon/2$ and then we'll be done. Now is the time to use the fact that f and g are continuous at a. For the epsilon we have, namely $\epsilon/2$, we know that there is a δ_f and δ_g such that when $0 < ||x - a|| < \delta_f$, then $||f(x) - f(a)|| < \epsilon/2$, and when $0 < ||x - a|| < \delta_g$, then $||g(x) - g(a)|| < \epsilon/2$. Now let $\delta = \min(\delta_f, \delta_g)$. Then when $0 < ||x - a|| < \delta$, then $||(f + g)(x) - (f + g)(a)|| \leq ||f(x) - f(a)|| + ||g(x) - g(a)|| < \epsilon/2 + \epsilon/2 = \epsilon$.

13. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a and $g : \mathbb{R}^m \to \mathbb{R}^p$ is continuous at $f(a)$, prove that the composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is continuous at a.

Answer: The philosophy is the same as the previous question. For a constant $\epsilon > 0$ and construct the $\delta > 0$ using what we know about f and g being continuous.

We'll play with the functions a bit first. I'll let $y = f(x)$ and $z = g(y)$, so $z = g \circ f(x) = g(f(x))$

Start with the given ϵ. Because g is continuous at $f(a)$, there is a δ_g such that when $0 < ||y - f(a)|| < \delta_g$, then $||g(y) - g(f(a))|| < \epsilon$. Now f is continuous at a. Hence for $\epsilon = \delta_g$, there is a δ_f such that when $0 < ||x - a|| < \delta_f$, then $||f(x) - f(a)|| < \epsilon = \delta_g$.

Let $\delta = \delta_f$ constructed above. Then when $0 < ||x - a|| < \delta$, then $||f(x) - f(a)|| < \delta_g$ and so $||g(f(x)) - g(f(a))|| < \epsilon$. But $||g(f(x)) - g(f(a))|| = ||g \circ f(x) - g \circ f(a)||$, hence $g \circ f$ is continuous at $x = a$.