Math 225 Homework 10

Deadline 5pm Friday May 1, 2009. No extensions given.

1. Find the determinant of the Jacobian of the following functions. Look familiar? (Hint: the trigonometric identity $\cos^2 A + \sin^2 A = 1$ will be very useful.)

   (a) $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$

   (b) $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $g(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$.

2. Find the image of the disk $u^2 + v^2 \leq 1$ under the transformation $x = au$, $y = bv$.

3. Use the transformation $x = 2u+v$, $y = u+2v$ to evaluate the integral $\iint_R (x-3y)dA$, where R is the triangular region with vertices $(0,0)$, $(2,1)$ and $(1,2)$.

4. Find the area of the ellipse $9x^2 + 4y^2 = 1$. Do this by using an appropriate change of variables. (Hint: What is the area of a circle of radius 1?)

5. Evaluate $\iint_R \frac{x-2y}{3x-y}\, dA$ where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, $3x - y = 8$. Do this by using an appropriate change of variables.

6. *Proof of Clairut's Theorem*

   Clairut's Theorem states that if a function $f : \mathbb{R}^n \to \mathbb{R}$ has $\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})$ and $\frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i})$ continuous in an open set containing a point a, then $\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})(a) = \frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})(a)$.

   In this question, we'll prove Clairut's Theorem using Fubini's Theorem and the Fundamental Theorem of Calculus.

   Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ and that $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$ and $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$ are continuous on the rectangle $A = [x_0, x_1] \times [y_0, y_1]$.

   (a) Compute $\int_A \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \int_{x_0}^{x_1} \left(\int_{y_0}^{y_1} \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\, dx \right) dy$.

   (b) Compute $\int_A \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \int_{y_0}^{y_1} \left(\int_{x_0}^{x_1} \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\, dy \right) dx$.

   (c) Compare your answers to (a) and (b) and you'll see $\int_A \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \int_A \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$. We now claim that this means $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$ for all points $(x, y) \in A$. Prove this claim in the following way:

By way of contradiction assume that there is a point $(a,b) \in A$ such that $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(a,b) \neq \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(a,b)$. Without loss of generality assume $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) > \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$.

i. Show that there is an open set containing (a,b) in A with $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) > \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$. (Hint: This is a one line proof.)

ii. Compute $\int_A \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) - \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$. What do you notice?

7. Find a parametrization for the line between points $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_3)$ in $\mathbb{R}^n$. (Explain how you got it.)

8. Evaluate the following line integrals.

   (a) $\int_C xy^3 ds$, where C is the curve $x = 4\sin t$, $y = 4\cos t$, $z = 3t$ from $t = 0$ to $t = \pi/2$.

   (b) $\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$, where C is consists of line segments from $(1,0,1)$ to $(2,3,1)$ and from $(2,3,1)$ to $(2,5,2)$.

   (c) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ and C is given by $\mathbf{r}(t) = 2t\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k}$, $-1 \leq t \leq 1$.

9. Determine whether or not $\mathbf{F}$ is conservative vector field. (That is, whether $\mathbf{F} = \nabla f$ for a function f.) If it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x,y) = xe^y\mathbf{i} + ye^x\mathbf{j}$

   (b) $\mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$

10. Find a function f such that $\mathbf{F} = \nabla f$, then use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Here, $\mathbf{F}(x,y) = y\mathbf{i} + (x+2y)\mathbf{j}$ and C is the upper semicircle that starts at $(0,1)$ and ends at $(2,1)$.

11. Show that the line integral $\int_C \tan y\,dx + x\sec^2 y\,dy$ is independent of path and then evaluate the integral. Here C is any path from $(1,0)$ to $(2,\pi/4)$. (Note: you'll need to be careful about the region that the path is in.)