

Math 225 Homework 1

Solutions

A grade of "A" on this HW is 40/48 and above. Note that 48 = 24+12+12 consists of problems + resources +meet & greet.

1 Problems

1. Problem 1.1 page 4 Spivak. Prove that

$$\|x\| \leq \sum_{i=1}^n \|x_i\|$$

Solution 1: . Observe that $x = (x_1, x_2, \dots, x_n) = (x_1, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \dots + (0, \dots, 0, x_n)$ and $\|(0, \dots, x_i, 0, \dots, 0)\| = \|x_i\|$. Using the triangle inequality

$$\|x\| \leq \|(x_1, 0, \dots, 0)\| + \|(0, x_2, 0, \dots, 0)\| + \dots + \|(0, \dots, 0, x_n)\| = \|x_1\| + \|x_2\| + \dots + \|x_n\|.$$

Solution 2: $\|x\| = \sqrt{(x_1)^2 + (x_2)^2 + \dots + (x_n)^2}$.
 $\sum_{i=1}^n \|x_i\| = \sum_{i=1}^n \sqrt{(x_i)^2} = \sqrt{(x_1)^2 + \dots + (x_n)^2}$. Note that this is **not** the same as $x_1 + x_2 + \dots + x_n$. To see this consider $x_i = -3!$ Square both sides so that $\|x\|^2 = (x_1)^2 + (x_2)^2 + \dots + (x_n)^2$ and $(\sqrt{(x_1)^2 + \dots + (x_n)^2})^2 = (x_1)^2 + (x_2)^2 + \dots + (x_n)^2 + C$. Here C is positive as each term is a positive square root. The desired inequality following immediately.

2. Problem 1.2 page 4 Spivak.

Solution: Where does the equality occur in $\|x + y\| \leq \|x\| + \|y\|$? If you look at the proof, it appears that the answer should be when x and y are linearly dependent. This is not the case. Let $x = ky$ for k a constant (this is the definition of x and y linearly dependent. Then $\|x + y\| = \|ky + y\| = \|y(k + 1)\| = |k + 1| \cdot \|y\|$. Now $\|x\| + \|y\| = \|ky\| + \|y\| = |k| \cdot \|y\| + \|y\| = \|y\|(1 + |k|)$. We want to know when there is equality, namely when is $|k + 1| = 1 + |k|$. Clearly this occurs when $k \geq 0$.

3. Problem 1.3 page 4 Spivak. (Hint: use the properties of norm found in Theorem 1-1.)

Solution: $\|x - y\| = \|x + (-y)\| \leq \|x\| + \|-y\| = \|x\| + |-1| \cdot \|y\| = \|x\| + \|y\|$. Equality occurs when $k \leq 0$. To see this let $x = ky$, then $\|x - y\| = \|ky - y\| =$

$\|y(k-1)\| = |k-1|\|y\|$. Now $\|x\| + \|y\| = \|ky\| + \|y\| = |k|\|y\| + \|y\| = \|y\|(1+|k|)$. Equality holds when $|k-1| = |-(1-k)| = 1+|k|$, that is when $k \leq 0$.

4. Problem 1.5 page 4 Spivak. (Hint: use the properties of norm found in Theorem 1-1.)

Solution: $\|y-x\|$ is the distance between y and x . Now $\|x-z\| = \|x-y+y-z\| = \|(x-y)+(y-z)\| \leq \|x-y\| + \|y-z\|$. This is called the triangle inequality. Why? Let ABC be a triangle and the length of the edge BC be a , AC be b and AB be c (draw a picture if you are confused). The triangle inequality states that the length of one side is less than or equal to the sum of the lengths of the other sides, or $c \leq a+b$. Now x, y and z are vectors in \mathbb{R}^n with tails at the origin and tips at points in \mathbb{R}^n . So the tips of x, y and z are the vertices of the triangle and $\|x-z\|$ is the length of the side of the triangle between the vertices at x and z . The geometric interpretation is now clear (I hope)!

5. A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called **norm preserving** if $\|T(x)\| = \|x\|$ and **inner-product preserving** if $\langle T(x), T(y) \rangle = \langle x, y \rangle$. (Here we assume that x, y are vectors in \mathbb{R}^n .)

- (a) Assume that T is norm preserving and show it must be inner-product preserving as well. (Hint: Look at the properties of inner-product in Theorem 1-2.)

Solution: Assume that $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$, we want to show that $\langle x, y \rangle = \langle T(x), T(y) \rangle$ for all $x, y \in \mathbb{R}^n$. We'll use the assumption, the polarization identity and $T(x+y) = T(x) + T(y)$ (as T is linear) to get what we want.

$$\begin{aligned} \langle T(x), T(y) \rangle &= \frac{1}{4}(\|T(x) + T(y)\|^2 - \|T(x) - T(y)\|^2) \\ &= \frac{1}{4}(\|T(x+y)\|^2 - \|T(x-y)\|^2) \\ &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \\ &= \langle x, y \rangle \end{aligned}$$

- (b) Assume that T is inner-product preserving and show it must be norm preserving as well.

Solution: We assume that $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$. In particular this is true when $x = y$, so

$$\|T(x)\| = \sqrt{\langle T(x), T(x) \rangle} = \sqrt{\langle x, x \rangle} = \|x\|.$$

- (c) Prove that such a linear transformation T is one-to-one. (Hint: assume it is not and try and get a contradiction.)

Solution: To show that a function is 1-1, we must show that if $x \neq y$, then $T(x) \neq T(y)$. (We could also show that if $T(x) = T(y)$, then $x = y$.) Let's do a proof by contradiction, assuming that T is not 1-1, so there exists an x and y with $x \neq y$, and $T(x) = T(y)$. Then $0 = T(x) - T(y)$, so

$$\begin{aligned} 0 &= \|T(x) - T(y)\| \\ &= \|T(x - y)\| \\ &= \|x - y\| \end{aligned}$$

Now, recall that $\|a\| = 0$ if and only if $a = 0$. So we've proved that $x - y = 0$ or $x = y$. This is a contradiction to the assumption that $x \neq y$.

Remark: You may have seen these transformations referred to as **orthogonal transformations**.

6. If $x, y \in \mathbb{R}^n$ are non-zero, the **angle** between x and y is defined by

$$\angle(x, y) = \arccos\left(\frac{\langle x, y \rangle}{\|x\|\|y\|}\right).$$

The linear transformation T is **angle preserving** if

- T is one-to-one.
- For $x, y \neq 0$, we have $\angle(T(x), T(y)) = \angle(x, y)$.

(a) Prove that if T is norm preserving, then T is angle preserving. (Hint: you may want to refer to results proved previously on this HW.)

Solution: We've proved that if T is norm preserving, then T is both inner-product preserving and 1-1. Then

$$\angle(T(x), T(y)) = \arccos\left(\frac{\langle T(x), T(y) \rangle}{\|T(x)\|\|T(y)\|}\right) = \arccos\left(\frac{\langle x, y \rangle}{\|x\|\|y\|}\right) = \angle(x, y).$$

(b) If there is a basis x_1, \dots, x_n of \mathbb{R}^n and numbers $\lambda_1, \dots, \lambda_n$ such that $T(x_i) = \lambda_i x_i$, prove that T is angle preserving if and only if all the λ_i are equal. (Hint: Each $x \in \mathbb{R}^n$ may be written uniquely as $x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$, where c_i are constants in \mathbb{R} .)

Solution: Before we launch into the proof, let's think about what we know. Given any $x \in \mathbb{R}^n$, then $x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$. Using the assumption that $T(x_i) = \lambda_i x_i$ and T is linear we have

$$\begin{aligned} T(x) &= T(c_1 x_1 + c_2 x_2 + \dots + c_n x_n) \\ &= T(c_1 x_1) + T(c_2 x_2) + \dots + T(c_n x_n) \\ &= c_1 T(x_1) + c_2 T(x_2) + \dots + c_n T(x_n) \\ &= c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots + c_n \lambda_n x_n. \end{aligned}$$

Assume that $\lambda_i = \lambda$ for $i = 1, 2, \dots, n$. Then

$$\begin{aligned} T(x) &= c_1\lambda_1x_1 + c_2\lambda_2x_2 + \dots + c_n\lambda_nx_n \\ &= \lambda(c_1x_1 + c_2x_2 + \dots + c_nx_n) \\ &= \lambda x. \end{aligned}$$

Altogether we see

$$\begin{aligned} \angle(T(x), T(y)) &= \arccos\left(\frac{\langle T(x), T(y) \rangle}{\|T(x)\| \|T(y)\|}\right) \\ &= \arccos\left(\frac{\langle \lambda x, \lambda y \rangle}{\|\lambda x\| \|\lambda y\|}\right) \\ &= \arccos\left(\frac{|\lambda|^2 \langle x, y \rangle}{|\lambda|^2 \|x\| \|y\|}\right) \\ &= \angle(x, y). \end{aligned}$$

We also need to show that T is 1-1, we proved this earlier using the assumption that T is norm preserving. So assuming the λ_i are all equal implies that T is angle preserving.

We now need to show that T angle preserving implies that all λ_i are equal. You can prove this directly. I'll prove the contrapositive statement. Namely we assume that not all the λ_i are equal, and try to show that T is not angle preserving. By assumption, there is at least one $\lambda_j \neq \lambda$. This means that $T(x) \neq \lambda x$. Take $x = x_j + x_i$ $T(x) = \lambda_j x_j + \lambda x_i$.

$$\begin{aligned} \angle(T(x), T(x_i)) &= \arccos\left(\frac{\langle \lambda_j x_j + \lambda x_i, \lambda x_i \rangle}{\|\lambda_j x_j + \lambda x_i\| \|\lambda x_i\|}\right) \\ &= \arccos\left(\frac{\langle \lambda_j x_j, \lambda x_i \rangle + \langle \lambda x_i, \lambda x_i \rangle}{\|\lambda_j x_j + \lambda x_i\| \|\lambda x_i\|}\right) \\ &\neq \arccos\left(\frac{\langle x, x_i \rangle}{\|x\| \|x_i\|}\right) \\ &= \angle(x, x_i). \end{aligned}$$

(c) What are all angle preserving transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$?

Solution: Rotations, dilations and contractions and combinations of these.