Final Exam May 2009
Math 225 Section 1 Advanced Calculus

Instructor: Elizabeth Denne

NAME: ____

Instructions

- This is a take-home final exam for Math 225. You have from Friday May 1 through 4.45pm Friday May 8, 2009.
- **You may not discuss the exam with any person while you are taking it. You are expected to follow the Smith honor code while taking this exam.**
- The deadline to submit solutions is 4.45pm Friday May 8, 2009. Failure to submit by this time will result in a grade of zero! Exams may be submitted in person to my office (310 Burton Hall) or placed in a sealed envelope and put in the slot on my office door. I will also accept electronic submissions in .pdf format.
- This exam consists of 10 questions. You must do all the questions. (Note that some questions have several parts.) There is also one optional bonus question.
- This is an open book exam. You may use the textbook, class notes, calculators and computers. If you do use a text or source other than these, please reference them.
- **For full credit please show all of your work and explain your reasoning carefully.**

To make the job of grading easier please:

1. Write your name on your solutions and staple all pages together.
2. Write on one side of the paper and submit the problems in the order assigned.
3. Write neatly!

I will be available for consultations during the exam period. I can help you with the following sorts of problems:

1. You don't understand the question.
2. You are stuck on how to get started.
3. You would like to discuss some of the material covered in class or from the homework.
4. You have done a computation and want to check the result.

Note that I will give the same information out to everybody (especially for hints to get you started). I can also answer some questions by email (allow 12 hours for an answer).

I am available for consultation at the following times:

1. 2 – 3.30pm Sunday May 3
2. 10.30am – 12noon Monday May 4
3. 2 – 4pm Wednesday May 6
4. 2 – 5pm Friday May 8

1. (8=4+4 points) The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{x}{2} + 1 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

   (a) Show that f is continuous when $x \neq 2$.

   (b) Show that f is not continuous when $x = 2$.

2. (8 points) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at $x = a$. Prove that the product $fg : \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$.
   (Hint 1: Recall that $fg(x) := f(x)g(x)$.)
   (Hint 2: $fg(x) - fg(a) = f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)$.)

3. (8=4+4 points) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Prove that f is differentiable at $x = 0$ and that $f'(0) = 0$. *(Hint: L'Hopital's rule may be helpful.)*

   (b) Now prove that $f^{(i)}(0) = 0$ for all i. (This is the ith derivative of f at $x = 0$.)

   You have proved that f is infinitely differentiable at $x = 0$!

4. (6 points) If $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at a, then prove that the product fg is differentiable at a and $D(fg)(a) = g(a)Df(a) + f(a)Dg(a)$.
   (Hint: this is one of the results of Corollary 2-4 in Spivak. When writing your proof, you may find the Chain Rule and Spivak's Theorem 2-3 helpful.)

5. (6=3+3 points)

   (a) Let $\gamma : \mathbb{R} \to \mathbb{R}^5$ be defined by $\gamma(t) = (\sin t, 3\cos t, 2t^2, \pi t - t^2, 4)$. Find $D\gamma(\pi/4)$.

   (b) Let $g : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $g(u, v) = (3uv^2 - 4, u^2 + 2uv, 3(u + v)^5)$. Find $Dg(-1, 1)$.

6. (6=4+2 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(u, v) = (2 - u + v^2, uv - 5)$.

   (a) For which points in $\mathbb{R}^2$ is this function invertible? State any theorem you use and show clearly why it works.

   (b) For the points where f is invertible, find the derivative of the inverse function f^{-1}. (Note: Do **not** find f^{-1}, just find $Df^{-1}(x, y)$.)

7. (6 points) The ellipsoid E in $\mathbb{R}^3$ is given by $E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 3y^2 + 4z^2 = 1\}$. Prove that E is a smooth surface. State any theorem you use and show clearly why it works.

8. (10=2+3+5 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 2 & \text{if } y \geq 1, \\ -1 & \text{if } 0 \leq y < 1, \\ 1 & \text{if } y < 0. \end{cases}$$

Consider f restricted to the region $A = [-1, 3] \times [-1, 3]$ in the xy-plane.

(a) Where if f continuous on A? Where is f discontinuous on A? (No need to prove this formally, just tell me where.)

(b) Let $B = \{(x, y) \in A \mid f(x, y) \text{ is discontinuous}\}$. Show that B has measure 0.

(c) This means that f is integrable on A. Find $\int_A f$.

9. (10=4+6 points)

(a) Find an expression for the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are constants. Do this by using an appropriate change of variables. *(Hint: What is the area of the circle $x^2 + y^2 = 1$?)*

(b) Compute the line integral $\int_C (x + yz)\, dx + 2x\, dy + xyz\, dz$, where C consists of line segments from $(1, 0, 1)$ to $(2, 3, 1)$ and from $(2, 3, 1)$ to $(2, 5, 2)$.

10. (8=2+4+2 points) Let $\mathbf{F}(x, y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. (Here I'm using the notation: $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.)

(b) Show that $\int \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. (Hint: Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.)

(c) Do your answers to (a) and (b) contradict each other? Please explain.

11. **Bonus Question** (10 Points) **Only attempt this problem once you've completed the rest of the exam.**

Assume the statement of the Implicit Function Theorem and use it to deduce the Inverse Function Theorem. Please use the statements of the theorems given below. (Note: in class we deduced the implicit function theorem from the inverse function theorem. In this question you are going the other way.)

Implicit Function Theorem: Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable in an open set containing (a, b) and $f(a, b) = 0$. Let M be the $m \times m$ matrix whose (i, j)th entry is $\frac{\partial f^i}{\partial x^{n+j}}(a, b)$ for $1 \leq i, j \leq m$. If $\det M \neq 0$, there is an open set $A \subset \mathbb{R}^n$ containing a and an open set $B \subset \mathbb{R}^m$ containing b with the following property: for each $x \in A$ there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. The function g is differentiable.

Inverse Function Theorem: Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on an open set containing the point a and $\det Df(a) \neq 0$. Then there is an open set V containing a and an open set W containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable and for all $y \in W$ satisfies

$$Df^{-1}(y) = (Df(f^{-1}(y)))^{-1}.$$

(In other words, the Jacobian matrix of f^{-1} at y is the inverse of the Jacobian matrix of f at $f^{-1}(y)$.)