Solutions to Continuity Problems

Rita Corey and Kirby Russell and Elizabeth Denne.

These are Rita and Kirby's solutions that I've tex-ed up (and added in some extra comments).

1. What are the two main definitions of continuity (one involves limits, one does not)? Using either definition, prove that if a function f is continuous and a function g is continuous, then the composition of f and g is continuous.

   Answer: If $A \subset \mathbb{R}^n$, then $f : A \to \mathbb{R}^m$ is continuous if for every open set $U \subset \mathbb{R}^m$, there is some open set $V \subset \mathbb{R}^n$ such that $f^{-1}(U) = V \cap A$.

   Given $A \subset \mathbb{R}^n$, then $f : A \to \mathbb{R}^m$ is continuous at $a \in A$ if for all $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < ||x - a|| < \delta$, then $||f(x) - f(a)|| < \epsilon$.

   The solution to the composition question is found in the solutions to HW 2 question 13.

2. If a function is bounded, what does this imply about the function's continuity?

   Answer: In general nothing. For example take the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   This function is bounded but is not continuous anywhere!

   In Spivak you'll find that a bounded function is continuous at a if and only if the oscillation is zero. We did not really discuss oscillation in class, so this question is not the kind of question you'd find on the midterm exam.

3. Using epsilon and delta, show whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} x^2 + 2x & \text{if } x \leq 2 \\ x^3 - 5x & \text{if } x > 2 \end{cases}$$

   is continuous.

   Answer: The function is **not** continuous at the point $x = 2$. Observe that $f(2) = 2^2 + 4 = 8$ and for x in the interval $(2, \sqrt{5})$, $f(x) \leq 0$. Pick $\epsilon = 1/2$. For any $\delta > 0$, if $0 \leq ||x-2|| < \delta$, there are always values of x for which $f(x) < 0$. For these values of x, $||f(x) - f(2)|| = ||f(x) - 8|| > 8$. So $||f(x) - f(2)|| < 1/2$ is not true.

The function is continuous at all other points. This is because a sums and products of continuous functions are continuous and we know that f is such a function. I'll add an $\epsilon - \delta$ proof once I've completed solutions to HW 6.

4. Using either method, show whether $f(x, y) = 2x + 3y - x^3y^2$ is continuous.

   Answer: The function is continuous at all other points. This is because a sums and products of continuous functions are continuous and we know that f is such a function. I'll add an $\epsilon - \delta$ proof once I've completed solutions to HW 6.

5. If two functions $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are both continuous at $x = a$, prove that the function $f + g : \mathbb{R}^n \to \mathbb{R}^m$ is also continuous at $x = a$.

   Answer: The solution is found in the solutions to HW 2 question 12.