Math 211 Midterm Exam 2

Practice problems

1. Short answer (or proof) questions.

   (a) Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, what is the relationship between the dimension of $\mathbb{R}^m$, $\ker(T)$ and $\text{im}(T)$?

   (b) Give an example of a 2×3 matrix A with $\dim(\ker(A)) = 1$

   (c) If a 3×3 matrix A represents the projection onto a plane in $\mathbb{R}^3$, what is $\text{rank}(A)$?

   (d) For which value(s) of the constant k are the vectors $\vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} k \\ 1 \\ k \end{bmatrix}$ perpendicular?

2. The matrix $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 3 & -2 & 8 & 2 & 14 \end{bmatrix}$ has $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Answer the following questions, being sure to explain how you arrive at your answers.

   (a) Find a basis for $\ker(A)$.

   (b) Find a basis for $\text{im}(A)$.

   (c) Find a basis for $\ker(A^T)$

   (d) What is the rank of A?

   (e) Are the rows of A linearly independent?

3. Determine whether the following statements are true or false. You **do not** need to give reasons for your answers.

   (a) A subspace V of $\mathbb{R}^n$ which has dimension m contains at least m linearly independent vectors.

   (b) A 3×7 matrix A must have $\dim(\ker(A)) \leq 2$.

   (c) If A is an $n \times m$ matrix, then $\text{im}(A) = \text{im}(\text{rref}(A))$.

4. A basis for $\mathbb{R}^2$ is $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$.

(a) Is this an orthonormal basis?

(b) Given a vector $\vec{x} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$ find the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B}$ and write the coordinate vector $[\vec{x}]_{\mathcal{B}}$.

(c) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathcal{B}$ where $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

5. Let V be the subspace of $\mathbb{R}^4$ with basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 2 \\ 2 \end{bmatrix}.$$

(a) Apply Gram-Schmidt to this basis to find an orthonormal basis $(\vec{u}_1, \vec{u}_2)$ for V.

(b) From your answer to (a) check that $\vec{u}_1 \cdot \vec{u}_2 = 0$.

(c) Find the orthogonal projection $\vec{e}_4^{\,\|} = \text{proj}_V(\vec{e}_4)$ of the standard vector $\vec{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ onto V.

(d) Express $\vec{e}_4$ as the sum of a vector in V with a vector in $V^{\perp}$.

Answers:

1. (a) This is the rank-nullity theorem: $m = \dim(\ker A) + \dim(\text{im}A)$.

   (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

   (c) $rank(A) = \dim(\text{im}(A)) = 2$.

   (d) Want k such that $k + 5 + 4k = 0$, so $k = -1$.

2. (a) Basis of $\ker(A)$ is $\begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

   (b) Basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

   (c) We know that $(\text{im}(A))^{\perp} = \ker(A^T)$. But $\dim(\text{im}(A)) = 3$ that is $\text{im}(A) = \mathbb{R}^3$. So the only thing perpendicular to it is $\vec{0}$. Hence $\ker(A^T) = \vec{0}$.

   (d) $rank(A) = 3$.

   (e) Yes.

3. (a) False. (Contains at most m linearly independent vectors.

   (b) False $\dim(\ker(A)) \leq 3$.

   (c) False $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$