Midterm 1 review problems

Math 211: last modified 9/27/09

These questions are designed to help you study for the first midterm exam. There is a variety of length and difficulties. I've included how many points I'd assign to each question, just like it was an actual exam. The questions on the exam may be different to the ones you see here. It is a very good idea to go over the questions you have done on the homework and make sure you understand the basic concepts of each section. Also make sure you understand where you went wrong on any homework questions that were marked incorrect.

1 Problems

1. (a) (4 points) Compute the row-reduced echelon form (rref) of the matrix

$$\begin{bmatrix} 0 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 2 & 1 & | & 0 & 1 & 0 \\ 3 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

(b) (4 points) Your computation in (a) gives the inverse of a 3×3 matrix A. What is A and what is A^{-1}?

(c) (2 points) Use the matrices A and A^{-1} of (b) to solve the linear system represented by $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Check your work by verifying that the entries of $\vec{x}$ actually satisfy that linear system.

2. Let $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ be the following vectors in $\mathbb{R}^4$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 6 \\ 7 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

Let V be the linear subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

(a) (2 points) Write a matrix whose image is V.

(b) (5 points) Find a basis of V. (Hint: what is a basis?)

(c) (3 points) Is the vector $\begin{bmatrix} -1 \\ 4 \\ 3 \\ 2 \end{bmatrix}$ in V?

3. For each of the following 10 assertions write True if the assertion is *true* and False if the assertion is *false*. Each is worth 1 point. For this question only there is no need to justify your answers. (Of course you are doing this as a practice for the exam, so please make sure you could justify your answers if you were asked to.)

1. If the linear system $A\vec{x} = \vec{b}$ has a unique solution, then A must be a square matrix.
2. Reflection about the line $x + y = 1$ is a linear transformation of $\mathbb{R}^2$.
3. A linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ always has infinitely many solutions if $n > m$.
4. If A is a matrix such that $AAAAA = I_2$, then A is invertible.
5. If A in an invertible matrix and B is a matrix such that AB is a zero matrix, then B is a zero matrix.
6. If A and B are invertible $n \times n$ matrices, then $(A + B)(A - B) = A^2 - B^2$.
7. Suppose a linear system has coefficient matrix A and augmented matrix B. If the system is consistent, then $rank(A) = rank(B)$.
8. If the vector $\vec{u}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$, then $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.
9. If vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent, then vectors $\vec{v}_2, \vec{v}_3, \vec{v}_4$ must be linearly independent as well.
10. The set $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \text{ is an integer} \right\}$ (infinitely many vertical lines at $x = 0$, $x = \pm 1$, $x = \pm 2 \ldots$) is closed under vector addition.

4. (4 points) Given the linear system

$$\begin{aligned} x_1 - 7x_2 + 3x_3 - 2x_4 &= -3 \\ 2x_1 - 7x_3 + 5x_4 &= 7 \\ 2x_2 - 8x_3 - 7x_4 &= 8 \end{aligned}$$

(a) Write down the corresponding augmented matrix

(b) Write down the corresponding matrix equation $A\vec{x} = \vec{b}$.

(c) Multiply out your answer to part (b) using the "column" method of multiplication.

(d) Use your answer to (c) to find a solution of this system **without doing any calculations**.

5. Given the matrices

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 0 & 1 & 6 \\ 0 & 1 & 3 & -4 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 8 & 4 & 7 & 0 \\ 0 & 1 & 9 & 4 \\ 3 & 5 & 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}$$

compute the following if they are defined. When they are not defined give reasons why.

(a) $A - 5B$

(b) AB

(c) BC

(d) D^{-1}

6. (2 points) Find the reduced row echelon form of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ -3 & 1 & -3 & 0 \\ 2 & 0 & 2 & -2 \end{bmatrix}$$

7. (a) (3 points) Write down the general solution of the system with augmented matrix

$$\begin{bmatrix} 1 & 5 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

and then express it as a sum of vectors.

8. Determine whether or not the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ is invertible. If it is, then find the inverse.

9. (6 points) Determine whether or not the following subset S are subspaces of the given $\mathbb{R}^n$. You must give reasons for your answers.

(a) $S = \left\{ \begin{bmatrix} a - 2b \\ b - 2c \\ 5b + c \end{bmatrix} \ : \ a,\, b,\, c \text{ in } \mathbb{R} \right\}$, S a subset of $\mathbb{R}^3$.

(b) $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1 \right\}$, S a subset of $\mathbb{R}^2$.

10. Consider a solution $\vec{x}_1$ of the linear system $A\vec{x} = \vec{b}$. Justify (prove) the facts stated in parts (a) and (b)

(a) If $\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$.

(b) If $\vec{x}_2$ is another solution of the system $A\vec{x} = \vec{b}$, then $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

11. Here are some questions which require short answers.

(a) Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$ are the vectors $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ linearly independent as well? How can you tell?

(b) Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^4$. Find $rref \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$.

(c) Is it true that if vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then $\vec{w}$ must be a linear combination of $\vec{v}$ and $\vec{u}$? Why?

(d) Is is true that if $\vec{u}, \vec{v}, \vec{w}$ are in a subspace V of $\mathbb{R}^n$, then $2\vec{u} - 3\vec{v} + 4\vec{w}$ must be in V as well? Why?

12. Consider a square matrix A with $ker(A^2) = ker(A^3)$. In the homework exercises you proved that $ker(A^3) = ker(A^4)$. Is it also true that $ker(A^4) = ker(A^5)$? You must justify (prove) your answer!

2 Very brief answers

These are just meant to help you check to see if you are on the right track. They are not really sufficient for an exam.

1. (a) $\begin{bmatrix} 1 & 0 & 0 & | & 0 & -1/3 & 1/3 \\ 0 & 1 & 0 & | & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} -1/3 & 1/3 & \\ -1/2 & 1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. Check using substitution.

2. (a) $A = [\vec{v}_1 \vec{v}_2 \vec{v}_3] = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 6 & 2 \\ 5 & 7 & 1 \\ 2 & 3 & 1 \end{bmatrix}$. Once you know that $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$, then you can use $[\vec{v}_1 \vec{v}_2]$.

(b) A basis of a subspace is a collection of linearly independent vectors which span the subspace. The columns of A span $im(A) = V$. We need to remove the redundant vectors. Now rref$(A) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Hence $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$. The basis is just $(\vec{v}_1, \vec{v}_2)$.

(c) rref$\left(\begin{bmatrix} 1 & 1 & | & -1 \\ 4 & 6 & | & 4 \\ 5 & 7 & | & 3 \\ 2 & 3 & | & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & | & -5 \\ 0 & 1 & | & 4 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$, hence the vector is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ (which one?) and so in in V.

3. F F F T T F T F T T (questions about why? please ask!)

4. (a) $\begin{bmatrix} 1 & -7 & 3 & -2 & | & -3 \\ 2 & 0 & -7 & 5 & | & 7 \\ 0 & 2 & -8 & -7 & | & 8 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & -7 & 3 & -2 \\ 2 & 0 & -7 & 5 \\ 0 & 2 & -8 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 8 \end{bmatrix}$.

(c) $x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -7 \\ -8 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 8 \end{bmatrix}$.

(d) $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

5. (a) not defined (b) $\begin{bmatrix} 9 & 0 & -1 & 12 \\ -1 & 1 & 2 & -4 \end{bmatrix}$ (c) not defined (d) $\begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-5t \\ t \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

8. yes it is invertible. Find rref$[A \mid I_3]$. Check your answer by multiplying!

9. (a) Yes it is a subspace. Two ways to check this.

(I) Observe $S = \left\{ \begin{bmatrix} a-2b \\ b-2c \\ 5b+c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

Thus $S = im\left(\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 5 & 1 \end{bmatrix} \right)$. In class we proved that the image of an $n \times m$ matrix is a subspace of $\mathbb{R}^n$, hence S is a subspace of $\mathbb{R}^3$.

(II) Check the three properties of a subspace. $\vec{0} = \begin{bmatrix} 0-2\cdot 0 \\ 0-2\cdot 0 \\ 5\cdot 0+0 \end{bmatrix}$, hence is in S. Take $\vec{x}_1, \vec{x}_2$ in S. Then there are constants a_1, b_1, c_1 and a_2, b_2, c_2 such that $\vec{x}_1 = \begin{bmatrix} a_1-2b_1 \\ b_1-2c_1 \\ 5b_1+c_1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} a_2-2b_2 \\ b_2-2c_2 \\ 5b_2+c_2 \end{bmatrix}$. Then $\vec{x}_1+\vec{x}_2$ is also in S. Why? Because $\vec{x}_1+\vec{x}_2 = \begin{bmatrix} (a_1-2b_1)+(a_2-2b_2) \\ (b_1-2c_1)+(b_2-2c_2) \\ (5b_1+c_1)+(5b_2+c_2) \end{bmatrix} = \begin{bmatrix} (a_1+a_2)-2(b_1+b_2) \\ (b_1+b_2)-2(c_1+c_2) \\ 5(b_1+b_2)+(c_1+c_2) \end{bmatrix}$. Hence we have found constants. $(a_1+a_2, (b_1+b_2), (c_1+c_2)$ which satisfy the conditions. A similar argument will show that if $\vec{x}$ is in S, then $k\vec{x}$ is in S. Please ask if you need help with this.

(b) $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : -1 \le x \le 1 \text{ and } -1 \le y \le 1 \right\}$, S a subset of $\mathbb{R}^2$.

This is not a subset. It is not closed under addition, nor scalar multiplication. Take $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ in S, so that $-1 \le x \le 1$ and $-1 \le y \le 1$. Then $3\vec{v} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$. It is not necessarily the case that $-1 \le 3x \le 1$. Just take $x = 1/2$.

10. (a) We know that $A\vec{x}_h = \vec{0}$ and $A\vec{x}_1 = \vec{b}$. We want to show $A(\vec{x}_1 + \vec{x}_h) = \vec{b}$. Now $A(\vec{x}_1 + \vec{x}_h) = A(\vec{x}_1) + A(\vec{x}_h) = \vec{b} + \vec{0} = \vec{b}$ and we are done.

(b) We know that $A\vec{x}_1 = \vec{b}$ and $A\vec{x}_2 = \vec{b}$. We want to show $A(\vec{x}_2 - \vec{x}_1) = \vec{0}$. Now $A(\vec{x}_2 - \vec{x}_1) = A(\vec{x}_1) - A(\vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$ and we are done.

11. (a) Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$ are the vectors $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ linearly independent as well?

Yes. We know $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ means that $c_1 = 0 = c_2 = c_3$ is the only solution. Now assume $d_1\vec{v}_1 + d_2(\vec{v}_1 + \vec{v}_2) + d_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}$. To prove the vectors linearly independent, we must show that $d_1 = 0 = d_2 = d_3$ is the only solution. Now $\vec{0} = d_1\vec{v}_1 + d_2(\vec{v}_1 + \vec{v}_2) + d_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = (d_1 + d_2 + d_3)\vec{v}_1 + (d_2 + d_3)\vec{v}_2 + d_3\vec{v}_3$. Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent means $d_1 + d_2 + d_3 = 0$, $d_2 + d_3 = 0$ and $d_3 = 0$ are the only solutions. But as $d_3 = 0$, this implies that d_2 and d_1 are zero as well.

(b) Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^4$. Find rref $\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$.

From class we know that the $n \times m$ matrix whose columns are linearly independent must have rank m. Hence rref $\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(c) Is it true that if vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then $\vec{w}$ must be a linear combination of $\vec{v}$ and $\vec{u}$?

No. $\vec{u}$ and $\vec{v}$ may be parallel and $\vec{w}$ some other vector not parallel. Example $\vec{v} = \vec{e}_1, \vec{u} = 2\vec{e}_1$ and $\vec{w} = \vec{e}_2$.

(d) Is is true that if $\vec{u}, \vec{v}, \vec{w}$ are in a subspace V of $\mathbb{R}^n$, then $2\vec{u} - 3\vec{v} + 4\vec{w}$ must be in V as well?

Yes. A subspace is closed under scalar multiplication, hence $2\vec{u}$, $-3\vec{v}$ and $4\vec{w}$ are in V. A subspace is closed under addition, hence $2\vec{u} - 3\vec{v} + 4\vec{w}$ must be in V.

12. Consider a square matrix A with $ker(A^2) = ker(A^3)$. In the homework exercises you proved that $ker(A^3) = ker(A^4)$. Is it also true that $ker(A^4) = ker(A^5)$?

This is true. How do you prove it? Go look at the solutions to section 3.1 question 50. Your argument should mimic this proof very very closely. Please as if you need more help with this. I also discuss this problem in the "careful reasoning" handout found on the webpage.

These solutions are brief and in some cases you'll need to give more details on an exam or homework assignment. Questions? Don't hesitate to ask!