Midterm Exam I Math 211 Fall 09

Grade Distribution.

A, A-: 14 B+, B, B-: 13, C: 2, C-/D: 2.

(1) a) $D-2C$ not defined as they are different sized matrices

b) AD not defined A 2×3 D 2×3 # col's $A \neq$ # rows D

c) $BC = \begin{bmatrix} 6+1+3 & 2+1+2 & -4+2+1 \\ 9-1+6 & 3-1+4 & -6-2+2 \\ 3+0+6 & 1+0+4 & -2+0+2 \end{bmatrix} = \begin{bmatrix} 10 & 5 & -1 \\ 14 & 6 & -6 \\ 9 & 5 & 0 \end{bmatrix}$

d) A^2 not defined as # cols $A=3 \neq$ # rows $A=2$.

e) $AB = \begin{bmatrix} 6+3-1 & 3-1+0 & 3+2-2 \\ 8+0+1 & 4+0+0 & 4+0+2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 3 \\ 9 & 4 & 6 \end{bmatrix}$

(2) Augmented matrix

a) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & k & 4 & 6 \\ 1 & 2 & k+2 & 6 \end{array}\right] \begin{array}{l} \\ -I \\ -I \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & k-2 & 1 & 2 \\ 0 & 0 & k-1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1/k-2 & 2/k-2 \\ 0 & 0 & 1 & 2/k-1 \end{array}\right]$

of course the latter only works if $k-2 \neq 0$ and $k-1 \neq 0$ or $k \neq 2$ & $k \neq 1$.

Of course 1's on diagonal means rref $A = I_3$ which will give a unique solution - so unique solution $\Leftrightarrow$ $k \in \mathbb{R}$ & $k \neq 2, k \neq 1$.

b) If $k=1$ Rref of augmented matrix has a row $[0\ 0\ 0 \mid 2]$ so no solution

c) If $k=2$ rref of augmented matrix has a row $[0\ 0\ 0 \mid 0]$ $\rightarrow$ infinite # solutions.

③ a) $\begin{bmatrix} 2 & 3 \\ 1 & k \end{bmatrix}$ invertible $\Leftrightarrow$ $2 \cdot k - 3 \cdot 1 \neq 0$
or $k \neq 3/2$

b) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is <u>not</u> invertible. $\text{rref} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$
($\text{rref} A \neq I_3$ rank $A = 2 \neq 3$)

c) $T: \mathbb{R}^2 \to \mathbb{R}^2$ scaling by 3. This is invertible since for each $\vec{y} \in \mathbb{R}^2$, $T\left(\frac{\vec{y}}{3}\right) = 3 \cdot \frac{\vec{y}}{3} = \vec{y}$, $\frac{\vec{y}}{3}$ is the <u>unique</u> vector with this property.
$T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ is scaling by $\frac{1}{3}$.

④ a) $A\vec{x} = \vec{b}$ has infinitely many solutions $\Rightarrow$ rank $A < 4$
$\Rightarrow$ rref $[A \,\vdots\, \vec{b}]$ has a row $[0 \cdots 0 1 0]$
$A\vec{x} = \vec{c}$ can have either an infinite # solutions
(a rref$[A \,\vdots\, \vec{c}]$ has a row $[0 0 0 0 1 0]$)
or $A\vec{x} = \vec{c}$ has <u>no</u> solutions
(a rref$[A \,\vdots\, \vec{c}]$ has a row $[0 0 0 0 1 1]$).

b) Solve $A\vec{x} = \vec{0}$ by considering rref$[A | \vec{0}]$
Solve $B\vec{x} = \vec{0}$ — — — — rref $[B | \vec{0}]$
Because rref$A$ = rref$B$
Then rref$[A | \vec{0}]$ = rref $[B | \vec{0}]$
hence the solutions are the same.

⑤ a) $(A-B)(A+B) = A(A+B) - B(A+B)$
$= AA + AB - BA - BB$
$= A^2 - B^2$ <u>only</u> if $AB - BA = [0]$
But $AB \neq BA$ in general.

5b) $T\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2x_1 - x_2\\x_2 - 2x_1\end{bmatrix} = \begin{bmatrix}2x_1 - x_2\\-2x_1 + x_2\end{bmatrix} = x_1\begin{bmatrix}2\\-2\end{bmatrix} + x_2\begin{bmatrix}-1\\1\end{bmatrix}$

<u>1st proof</u>: $= \begin{bmatrix}2 & -1\\-2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ is a linear transformation since $T(\vec{x}) = A\vec{x}$ $A = \begin{bmatrix}2 & -1\\-2 & 1\end{bmatrix}$.

<u>Second proof</u>: $T\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0-0\\0-0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

- $T\begin{bmatrix}x_1 + y_1\\x_2 + y_2\end{bmatrix} = \begin{bmatrix}2(x_1+y_1) - (x_2+y_2)\\(x_2+y_2) - 2(x_1+y_1)\end{bmatrix} = \begin{bmatrix}(2x_1 - x_2) + (2y_1 - y_2)\\(x_2 - 2x_1) + (y_2 - 2y_1)\end{bmatrix}$
$= \begin{bmatrix}2x_1 - x_2\\x_2 - 2x_1\end{bmatrix} + \begin{bmatrix}2y_1 - y_2\\y_2 - 2y_1\end{bmatrix} = T\begin{bmatrix}x_1\\x_2\end{bmatrix} + T\begin{bmatrix}y_1\\y_2\end{bmatrix}$
- $T\begin{bmatrix}kx_1\\kx_2\end{bmatrix} = \begin{bmatrix}2kx_1 - kx_2\\kx_2 - 2kx_1\end{bmatrix} = k\begin{bmatrix}2x_1 - x_2\\x_2 - 2x_1\end{bmatrix} = kT\begin{bmatrix}x_1\\x_2\end{bmatrix}$

(6) a) False $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ is in rref.

b) True $\begin{bmatrix}3 & -4\\4 & 3\end{bmatrix}$ is a rotation combined with a scaling.

c) <u>False</u> $A^2 = I_n$ for $n\times n$ invertible matrix A. Then A must be I_n

$A = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ rotation by π

d) False $A\vec{x} = \vec{b}$ has unique solution Then A need not be square. [Ex. $A = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$ $\vec{b} = \begin{bmatrix}2\\3\\0\end{bmatrix}$ $\vec{x} = \begin{bmatrix}2\\3\\0\end{bmatrix}$ unique solution]

e) <u>False</u> If rank $A = 3$ & A is 3×3 then rref $A = I_3$ so $A\vec{x} = \vec{b}$ has a <u>unique</u> solution.

f) <u>True</u> $(AB)^{-1} = B^{-1}A^{-1}$ proved in class. (A, B $n\times n$ & invertible.)

g) <u>True</u> orthogonal projection onto a line L

$A\vec{x}$ is a vector $\vec{v}$, say, on L & $A\vec{v} = \vec{v}$ for all $\vec{v} \in L$

so $A^2\vec{x} = A(A\vec{x}) = A\vec{v} = \vec{v}$