Homework 9 Math 211

Due 4pm Friday November 6, 2009.

Section 3.3

1. In this exercise, find the redundant column vector of the given matrix A "by inspection". Then find a basis of the image of A and a basis of the kernel of A.

$$A = \begin{bmatrix} 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Find the reduced row echelon form of the matrix $A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{bmatrix}$. Then find a basis for the image of A and a basis for the kernel of A.

Section 3.4

1. In the exercises below, determine whether the vector $\vec{x}$ is in the span V of the vectors $\vec{v}_1, \ldots, \vec{v}_m$. (If you can just " see it" then that's OK, otherwise solve using rref as in examples 1 and 2.) If $\vec{x}$ is in V, find the coordinates of $\vec{x}$ with respect to the bases $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of V and write the coordinate vector $[\vec{x}]_{\mathcal{B}}$.

   (a) $\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (b) $\vec{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

   (c) $\vec{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$: $\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

   (d) $\vec{x} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$.

2. Consider the basis $\mathcal{B}$ of $\mathbb{R}^2$ consisting of vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. We are told that $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ for a certain vector $\vec{x}$. Find $\vec{x}$.

3. Consider the plane $x_1 + 2x_2 + 2x_3 = 0$ in $\mathbb{R}^3$ with basis $\mathcal{B}$ consisting of vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. If $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, find $\vec{x}$.

4. In the exercises below, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_m)$.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 4 & -2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

5. Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with respect to the basis $\mathcal{B} = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix})$. For practice, solve each problem in three ways:

   (a) Construct B "column by column"

   (b) Use the formula $B = S^{-1}AS$

   (c) Use a commutative diagram, as in examples 3 and 4.

6. If $\mathcal{B}$ is a basis of a subspace V of $\mathbb{R}^n$, then prove that $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$.

7. Prove that

   (a) An $n \times n$ matrix A is similar to itself

   (b) If A is similar to B, then B is similar to A.

8. If A is similar to B, what is relationship between rank(A) and rank(B)?

True/False

Please submit the answers to these questions on a separate page with your name on it.

Are the following statements True or False? You must give a reason for your answer to receive full credit.

1. There exists a 2×2 matrix A such that $\text{im}(A) = \ker(A)$.
2. If $A^2 = 0$ for a 10×10 matrix A, then the inequality $\text{rank}(A) \leq 5$ must hold.
3. If V is any three-dimensional subspace of $\mathbb{R}^5$, then V has infinitely many bases.
4. If A and B are $n \times n$ matrices and vector $\vec{v}$ is in the image of both A and B, then $\vec{v}$ must be in the image of matrix $A + B$ as well.
5. If an $n \times n$ matrix A is similar to B and B is similar to C, then C must be similar to A as well
6. If A is similar to B and A is invertible, then B must be invertible as well.
7. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.
8. If an $n \times n$ matrix A is similar to matrix B, then $A + 7I_n$ must be similar to $B + 7I_n$.

Optional Reading

Linear algebra has applications throughout mathematics, the sciences and social sciences. You now know enough to start applying your knowledge to your areas of interest. For example:

1. If you are interested in **Economics** read exercises 78 on page 150.