Homework 8 Math 211

Due 4pm Friday October 30, 2009.

Section 3.1

1. Give an example of a matrix A such that $\text{im}(A) = \text{span}(\begin{bmatrix} 1 \\ 5 \end{bmatrix})$.

2. Consider an $n \times p$ matrix A and a $p \times m$ matrix B. If $\ker(A) = \text{im}(B)$, what can you say about the product AB?

Section 3.2

1. Consider a 5×4 matrix $A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}$. We are told the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is in the kernel of A. Write $\vec{v}_4$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

2. Are the columns of an invertible matrix linearly independent? Why or why not?

3. Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^4$. Find $\text{rref} \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$.

4. Express the plane V in $\mathbb{R}^3$ with equation $3x_1 + 4x_2 + 5x_3 = 0$ as

   (a) the kernel of a matrix A, and

   (b) the image of a matrix B.

   (So in this question, I need you to tell me what the matrices are. You'll put in enough explanation so that it is clear why $V = \text{im}\, B$ and $V = \ker A$.)

Section 3.3

1. In this exercise, find the redundant column vector of the given matrix A "by inspection". Then find a basis of the image of A and a basis of the kernel of A.

   (a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2. Find the reduced row echelon form of the matrix $A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix}$. Then find a basis for the image of A and a basis for the kernel of A.

3. Determine whether the following vectors form a basis of $\mathbb{R}^4$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}.$$

4. Find a basis of the subspace of $\mathbb{R}^3$ defined by the equation $2x_1 + 3x_2 + x_3 = 0$.

5. Can you find a 3×3 matrix A, such that $\text{im}(A) = \ker(A)$?

6. Give an example of a 4×5 matrix A with $\dim(\ker(A)) = 3$.

7. Consider a linear transformation T from $\mathbb{R}^5$ to $\mathbb{R}^3$. What are the possible values of $\dim(\ker(T))$? Explain.

8. Consider a linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^7$. What are the possible values of $\dim(\text{im}(T))$? Explain.

True/False

Please submit the answers to these questions on a separate page with your name on it.

Are the following statements True or False? You must give a reason for your answer to receive full credit.

1. The vectors of the form $\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix}$ (where a and b are arbitrary real numbers) form a subspace of $\mathbb{R}^4$.

2. If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.

3. If the kernel of a matrix A consists of the zero vector only, then the column vectors of A must be linearly independent.

4. The column vectors of a 5×4 matrix must be linearly dependent.

5. If vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span $\mathbb{R}^4$, then n must be equal to 4.

6. If a subspace V of $\mathbb{R}^3$ contains the standard vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then V must be $\mathbb{R}^3$.

7. If A is any $n \times n$ matrix A such that $A^2 = A$, then the image of A and the kernel of A have only the zero vector in common.

8. If A is an invertible $n \times n$ matrix, then the kernels of A and A^{-1} must be equal.

Optional Extra Practice

1. Consider a linear transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$ and some linearly independent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^m$. Are the vectors $T(\vec{v}_1)$, $T(\vec{v}_2), \ldots, T(\vec{v}_p)$ linearly independent or linearly dependent? Explain.

2. Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$. Are the vectors $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ linearly independent as well? How can you tell?

3. Let V be the subspace of $\mathbb{R}^4$ defined by the equation $x_1 - x_2 + 2x_3 + 4x_4 = 0$. Find a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^4$ such that $\ker(T) = \{\vec{0}\}$ and $\text{im}(T) = V$. Describe T by its matrix A¿

4. Consider the matrices $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$,

$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) Which of the matrices in this list have the same kernel as matrix C?

(b) Which of the matrices in this list have the same image as matrix C?

(c) Which of these matrices has an image that is different from the images of all the other matrices in the list?