Solutions HW6 Math 211

$\boxed{\text{Section 2.4}}$

1. A invertible, c nonzero scalar.
Matrix cA is invertible with inverse $\frac{1}{c}A^{-1}$

<u>Proof</u> $(cA)(\frac{1}{c}A^{-1}) = (c\frac{1}{c})(AA^{-1}) = I_n$

The criterion for invertibility then says that cA is invertible & $\frac{1}{c}A^{-1}$ is its inverse.

2. A, B invertible $n \times n$ matrices.

a) $(A+B)^2 = A^2 + 2AB + B^2$ is <u>not</u> necessarily true.

$(A+B)(A+B) = (A+B)A + (A+B)B$
$= AA + BA + AB + BB = A^2 + BA + AB + B^2$

It is not true that $AB = BA$ for all matrices.

b) A^2 invertible and $(A^2)^{-1} = (A^{-1})^2$.
In class we proved that if A, B are invertible, then so is AB & $(AB)^{-1} = B^{-1}A^{-1}$. Apply this result with $B = A$, to give $AA = A^2$ invertible & $(A^2)^{-1} = A^{-1}A^{-1} = (A^{-1})^2$.

c) $A+B$ is not necessarily invertible.
Example $I_n + (-1)I_n = 0$ matrix
Even if $A+B$ is invertible, $(A+B)^{-1}$ does <u>not</u> necessarily equal $A^{-1} + B^{-1}$. Example $I_n + I_n = 2I_n$ & $(2I_n)^{-1} = \frac{1}{2}I_n$
But $\frac{1}{2}I_n \neq I_n^{-1} + I_n^{-1} = 2I_n$.

d) $(A-B)(A+B) = A^2 - BA + AB - B^2 \neq A^2 - B^2$ in general since $AB \neq BA$ in general.

e) $A(BB^{-1})A^{-1} = A(I_nA^{-1}) = AA^{-1} = I_n$. TRUE.

f) $ABA^{-1} \neq B$ in general.
If $ABA^{-1} = B$, then $ABA^{-1}A = BA$ or $AB = BA$.
But we know $AB \neq BA$ in general.

g) $(ABA^{-1})^3 = (ABA^{-1})(ABA^{-1})(ABA^{-1})$
$= AB(A^{-1}A)B(A^{-1}A)BA^{-1}$ TRUE
$= A(BI_n)(BI_n)BA^{-1} = ABBBA^{-1} = AB^3A^{-1}$

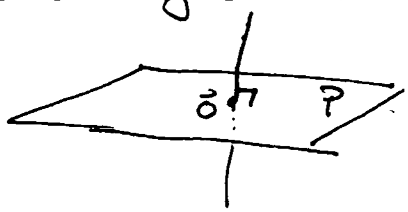
h) $(I_n+A)(I_n+A^{-1}) = I_nI_n + AI_n + I_nA^{-1} + AA^{-1}$ TRUE
$= I_n + A + A^{-1} + I_n = 2I_n + A + A^{-1}$

i) $A^{-1}B$ is invertible, since A^{-1} & B are invertible.
$(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A.$ TRUE.

$\boxed{\text{Section 3.1.}}$

1a) Reflection about $y=2x$ in $\mathbb{R}^2$.
Image all of $\mathbb{R}^2$ (
Kernel $\{\vec{0}\}$ (All nonzero vectors stay nonzero under reflection)

b) orthogonal projection onto plane $x+2y+3z=0$
Image is the plane $x+2y+3z=0$
kernel is the line perpendicular to the plane through the origin.



c) Rotation through $\pi/3$ ↺ in $\mathbb{R}^2$
Image all of $\mathbb{R}^2$. Kernel $\{\vec{0}\}$ (only thing that is fixed.)

2.a) $\ker\begin{bmatrix}2&3\\6&9\end{bmatrix}$ solve $A\vec{x}=\vec{0}$

$$\left[\begin{array}{cc|c}2&3&0\\6&9&0\end{array}\right] \to \left[\begin{array}{cc|c}2&3&0\\0&0&0\end{array}\right] \to \left[\begin{array}{cc|c}1&3/2&0\\0&0&0\end{array}\right]$$

$x_1 = -\frac{3}{2}x_2$ $\vec{x} = \begin{bmatrix}-\frac{3}{2}t\\t\end{bmatrix}$
x_2 is free.

$\ker A = \left\{\vec{x} = t\begin{bmatrix}-3/2\\1\end{bmatrix}\right\} = \text{span}\begin{bmatrix}-3/2\\1\end{bmatrix}$

b) $\ker[1\ 2\ 3]$ $\quad [1\ 2\ 3\,|\,0]$ $\quad$ x_2, x_3 free $x_1+2x_2+3x_3=0$

$\vec{x} = \begin{bmatrix}-2s-3t\\s\\t\end{bmatrix} = s\begin{bmatrix}-2\\1\\0\end{bmatrix} + t\begin{bmatrix}-3\\0\\1\end{bmatrix}$ $\quad \ker A = \text{span}\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right)$

c) $\ker A$ solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_3 free.

$x_1 = -s - t$
$x_2 = s$
$x_3 = t$

$$\vec{x} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \ker A = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

d)
$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$
$x_2 = 0$
~~x_3 free.~~

$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ solution so $\ker A = \{\vec{0}\} = \text{span}(\vec{0})$

3 a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ $\quad \text{im} A = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

(or any other pair of column vectors.)
$\text{Im} A = \text{span}(\text{col. vectors of } A)$ & is a subspace of $\mathbb{R}^2$.
But $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for any k, so the span is $\mathbb{R}^2$.

b) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ $\quad \text{Im} A = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right)$ (or any of the other column vectors.)

Each column vector is a multiple of the others, so only one is needed to describe the span.

c) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\quad \text{Im} A = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$

Both column vectors are needed to describe the image as one is <u>not</u> a linear combination of the other.

T/F Solutions 211 HW 6.

1. There is an invertible 2×2 matrix A s.t. $A^{-1} = -A$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

TRUE

Without loss of generality assume $ad-bc=1$.
(Indeed we must assume this if $\frac{b}{ad-bc} = b$, for example.)

Then we see $d=-a$, b, c are ~~free~~. But $ad-bc=1 \Rightarrow$
$a(-a)-bc=1$ or $-a^2 = 1+bc$ or $a^2 = -bc-1$
so $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has inverse $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

2. $\boxed{\text{FALSE}}$ Let $B = I_n$.
Then $AI_n = I_nA = A$, $CI_n = I_nC = C$, But $AC \neq CA$ in general.

3. $\boxed{\text{FALSE}}$ A 10×10 matrix with 92 ones, must have at least two rows consisting only of ones. This means rref(A) has a row of zeros, so A is not invertible.

4. $\boxed{\text{FALSE}}$ $A=B=I_n$ $B=-I_n$ Then $A+B=0$ matrix which is not invertible.

5. $\boxed{\text{TRUE}}$ If A^2 is invertible, then $A^2B = I_n$ for a matrix B, where $B = (A^2)^{-1}$.
Rearranging, we see $A(AB) = I_n$. Hence A is invertible by the criterion for invertibility.
(Note that you cannot argue that A is invertible because $(A^2)^{-1} = (A^{-1})^2$. This preassumes that A is invertible!).