Homework 3 Math 211

Due 4pm September 25, 2009.

Section 2.1

1. Compute the products $A\vec{x}$ in parts (a), (b), (c) below using paper and pencil. In each case, compute the product two ways: in terms of the columns of A (Definition 1.3.8) and in terms of the rows of A (Fact 1.3.7).

   (a) $\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 5 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$

2. Consider the transformations from $\mathbb{R}^2 \to \mathbb{R}^3$ defined below. Which of these transformations are linear? Explain.

   (a) $\begin{bmatrix} y_1 = 2x_2 \\ y_2 = x_2 + 2 \\ y_3 = 2x_2 \end{bmatrix}$

   (b) $\begin{bmatrix} y_1 = 2x_2 \\ y_2 = 3x_2 \\ y_3 = x_1 \end{bmatrix}$

   (c) $\begin{bmatrix} y_1 = x_2 - x_3 \\ y_2 = x_1 x_3 \\ y_3 = x_1 - x_2 \end{bmatrix}$

3. Find the matrix of the linear transformation $\begin{bmatrix} y_1 = 9x_1 + 3x_2 - 3x_3 \\ y_2 = 2x_1 - 9x_2 + x_3 \\ y_3 = 4x_1 - 9x_2 - 2x_3 \\ y_4 = 5x_1 + x_2 + 5x_3 \end{bmatrix}$.

4. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}4\\9\end{bmatrix}, \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-13\\7\end{bmatrix}.$$

Find the matrix of T.

5. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\3\\-2\end{bmatrix} + x_2\begin{bmatrix}4\\0\\3\end{bmatrix}.$$

Is this transformation linear? If so, find its matrix.

6. Give a geometric interpretation of the linear transformations defined by the matrices below. Show the effect of these matrices on the letter L from Example 5 in Section 2.1 of the text.

(a) $\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$

(b) $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

(c) $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

(d) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

(e) $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$

7. Show that if T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$T\begin{bmatrix}x_1\\x_2\\\vdots\\x_m\end{bmatrix} = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_mT(\vec{e}_m),$$

where $\vec{e}_1$, $\vec{e}_2$,…, $\vec{e}_m$ are the standard vectors in $\mathbb{R}^n$.

8. Describe all linear transformations from $\mathbb{R}$ to $\mathbb{R}$.

9. (a) Consider the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ (the dot product) from $\mathbb{R}^3$ to $\mathbb{R}$ linear? If so, find the matrix of T.

   (b) Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ linear? If so, find the matrix of T (in terms of the components of $\vec{v}$.)

   (c) Conversely, consider a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$. Show there exists a vector $\vec{v}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{v} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.

True/False

Please submit the answers to these questions on a separate page with your name on it.
Are the following statements True or False? You must give a reason for your answer to receive full credit.

1. If vector $\vec{u}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$, then $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.

2. A linear system with fewer unknowns than equations must have infinitely many solutions or none.

3. If the system $A\vec{x} = \vec{b}$ has a unique solution, then A must be a square matrix.

4. Vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

5. The system $A\vec{x} = \vec{b}$ is inconsistent if and only if $\text{rref}(A)$ contains a row of zeros

6. If matrix E is in reduced row-echelon form and if we omit a column of E, then the remaining matrix must be in reduced row-echelon form as well.

Extra practice - do NOT submit for grading.

1. Find solutions to the following linear systems using Gauss-Jordan elimination as discussed in Section 1.2 of the text. Show all your work.

$$\begin{bmatrix} x_4 + 2x_5 - x_6 = 2 \\ x_1 + 2x_2 + x_5 - x_6 = 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 = 2 \end{bmatrix}$$

2. Consider the equations $\begin{bmatrix} y + 2kz = 0 \\ x + 2y + 6z = 2 \\ kx + 2z = 1 \end{bmatrix}$, where k is an arbitrary constant.

   (a) For which values of k does this system have a unique solution?

   (b) When is there no solutions?

   (c) When are there infinitely many solutions?