Criterion for Invertibility

Math 211

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation represented by an $n \times n$ matrix A. Then A (or T) is invertible if any of the following are true

- For each $\vec{y}$ in $\mathbb{R}^n$ there is a unique $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{y}$.
- If $A\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.
- $rref(A) = I_n$
- $rank(A) = n$
- $im(A) = \mathbb{R}^n$
- $ker(A) = \{0\}$.

Theorem 1. *(Criterion for Invertibility) Let A and B be two $n \times n$ matrices such that*

$$BA = I_n.$$

Then

- *A and B are both invertible,*
- *$A^{-1} = B$ and $B^{-1} = A$,*
- *$AB = I_n$.*

Proof. Let's first show A is invertible. To do this, we just need to show that if $A\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.

Let's multiply both sides by B, so $B(A\vec{x}) = B\vec{0} = \vec{0}$. However, $BA = I_n$. Altogether this means $BA\vec{x} = I_n\vec{x} = \vec{0}$. So $\vec{x} = \vec{0}$ and A is invertible.

Multiply both sides of $BA = I_n$ by A^{-1} on the right, so $BAA^{-1} = I_nA^{-1}$ and $B = A^{-1}$. Now B **is** A^{-1} and we know this is invertible. Hence $B^{-1} = (A^{-1})^{-1} = A$.

Finally $AB = AA^{-1} = I_n$.

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