Calculus III: vectors worksheet

In questions 1 through 12, let

$$A = (1,1), \qquad B = (2,-1), \qquad C = (0,5),$$
$$D = (-1,6), \qquad E = (0,0), \qquad F = (\sqrt{3}, \pi).$$

1. Using a single coordinate plane $\mathbb{R}^2$ for each part (a), (b), (c), (d), sketch *and label* each of the following sets of points (vectors):
   (a) A, B, C, D, E, F.
   (b) $3A$, $A+C$, $B-D$, $3A+D$, $E+F$, $2B+C$.
   (c) $A+B$, $A+2B$, $A+3B$, $A+\frac{1}{2}B$, $A-B$, $A+\sqrt{2}\,B$.
   (d) The point midway between the points B and D.
2. Determine the six lengths $\|A\|, \ldots, \|F\|$.
3. The vector B displaces the point $(5,7)$ to _____ and the point $(-9,-4)$ to _____.
4. What point is displaced by the vector B to the point $(1,3)$? What point is displaced by B to $(0,0)$?
5. What vector displaces the point $(4,-1)$ to the point $(3,3)$? What vector displaces the point D to the point C?
6. The vector that displaces $(-3,2)$ to $(6,3)$ also displaces $(-8,-6)$ to _____ and $(6,3)$ to _____.
7. Show that the point midway between B and D is $\frac{1}{2}B + \frac{1}{2}D$.
8. Find the point on the line from D to A that is one-quarter of the way from D to A; find the point that is *three*-quarters of the way.
9. **Triangle inequality** In Euclidean geometry, the sum of the lengths of any two sides of a triangle is greater than the length of the third side.
   (a) Draw the triangle whose sides correspond to the vectors A, B, and $A+B$.
   (b) Compute $\|A+B\|$ to demonstrate $\|A+B\| < \|A\| + \|B\|$ (thus verifying the triangle inequality in this case, at least).
   (c) Find non-zero vectors X and Y for which $\|X+Y\| = \|X\| + \|Y\|$.
   (d) Does this contradict the triangle inequality? In fact, the "triangle" with sides X, Y, and $X+Y$ has a special (i.e., unusual) shape; what does it look like?

The last example leads us to reformulate the **triangle inequality** in the following, more inclusive, way. The sum of the lengths of any two sides of a triangle is greater than *or equal to* the length of the third side: for any vectors P and Q, $\|P+Q\| \le \|P\| + \|Q\|$.

10. Sketch in the coordinate plane the set of all points of the form $A+tB$ when $-1 \le t \le 3$. (Note this is an extension of question 1c.)

11. (a) Describe the set $\mathcal{L}_1$ of all points of the form $A+tB$, when t varies over the set $\mathbb{R}$ of all real numbers.

    (b) Describe the set $\mathcal{L}_2$ of all points of the form tB, when t varies over the set $\mathbb{R}$ of all real numbers.

    (c) How are the sets $\mathcal{L}_1$ and $\mathcal{L}_2$ related?

12. For each of the following expressions, either compute it or explain why it does not make sense:

$$A \cdot B \qquad (F+C)\cdot B \qquad (F\cdot C)+B \qquad -C \qquad \|3A\| \qquad \|-C\|$$
$$\frac{A}{3} \qquad \frac{B}{C} \qquad \|3C-B\| \qquad E\cdot E \qquad \sqrt{F\cdot F} \qquad \sqrt{B\cdot D}$$

13. Let $X=(x,y)$ and $\lambda \in \mathbb{R}$. Show (e.g., by providing an example) that the statement $\|\lambda X\| = \lambda\,\|X\|$ is *not* true in general. Modify the statement to describe the true relation between $\|\lambda X\|$ and λ and $\|X\|$.

14. What is the cosine of the angle between $A=(5,-2)$ and C, where (a) $C=(-1,7)$; (b) $C=(-7,1)$; (c) $C=(-3,21)$; (d) $C=(10,-4)$?

15. Sketch $P=(5,-2)$ and $Q=(4,10)$ in $\mathbb{R}^2$ and compute $P\cdot Q$. Is $P \perp Q$?

16. Find three different vectors Z for which $Z\cdot T = S\cdot T$ when

    (a) $S=(4,3)$ and $T=(2,0)$;

    (b) $S=(4,3)$ and $T=(1,5)$.

17. Suppose U, V and W are non-zero vectors in $\mathbb{R}^2$ for which $U\cdot W = V\cdot W$; then $U-V \perp W$. Explain why.

18. The vector that displaces $(5,-1,3)$ to $(4,3,1)$ also displaces $(4,3,1)$ to _____ and $(0,0,0)$ to _____.

19. If the vector T displaces $(2,-1,3)$ to $(0,1,1)$, then $4T$ displaces $(2,-1,3)$ to _____ and $(0,0,0)$ to _____.

20. Suppose the vector R displaces $(1,2,3)$ to $(0,-3,5)$ and S displaces $(1,2,3)$ to $(7,0,9)$.

    (a) How does $R+S$ displace $(1,2,3)$?

(b) What point does $R + S$ displace to $(0, 0, 0)$?

(c) How does $5R$ displace $(1, 2, 3)$?

(d) How does $-2S$ displace $(7, 0, 9)$?

(e) How does $5R - 2S$ displace $(1, 1, 1)$?

(f) Find values of α and β for which the vector $\alpha R + \beta S$ displaces $(0, 0, 0)$ to $(1, 1, 1)$ *if this is possible*, or else explain why it is impossible.

(g) Determine $\|R\|$ and $\|S\|$.

21. Calculate both the dot product $A \cdot C$ and the cross product $A \times C$ when

(a) $A = (1, 2, 3)$, $C = (4, 5, 6)$;

(b) $A = (4, 5, 6)$, $C = (1, 2, 3)$;

(c) $A = (1, 2, 3)$, $C = (4, 8, 12)$;

(d) $A = (1, 2, 3)$, $C = (2, -1, 0)$;

(e) $A = (1, 0, 0)$, $C = (0, 0, 1)$;

22. Calculate the length of each of the vectors (i.e., A, C, and $A \times C$) in each part of the preceding question.

23. Let $A = (a, b, c)$, $P = (p, q, r)$.

(a) Is $A \cdot P = P \cdot A$? Compute these to decide the question.

(b) Is $A \times P = P \times A$?

(c) Calculate $A \cdot (A \times P)$ and $P \cdot (A \times P)$. What does this result tell you about how $A \times P$ is related geometrically to A and P?

(d) Calculate $A \times A$ and $A \times kA$, where k is an arbitrary scalar.

24. In each of the following, sketch the vectors A and C and the projection of C on (the line in the direction of) A, then calculate the *signed* length of that projection.

(a) $A = (6, 0)$, $C = (3, 4)$.

(b) $A = (6, 0)$, $C = (-3, 4)$.

(c) $A = (3, 4)$, $C = (6, 0)$.

(d) $A = (1, 1)$, $C = (3, -4)$.

(e) $A = (6, -2)$, $C = (1, 3)$.

25. For each pair of vectors A, C in the previous question, let $\mathcal{L}_A$ be the line through the origin perpendicular to A. Determine whether C is on the same side of $\mathcal{L}_A$ as A, on the opposite side, or on $\mathcal{L}_A$ itself.

26. (a) Each of the *ordered* pairs of vectors A, C in the question 24 determines an *oriented* parallelogram. Indicate, in each case, whether the orientation is positive or negative.

    (b) Calculate the signed area of each of these oriented parallelograms. In each case, the area is the determinant of a certain 2×2 matrix; write out that matrix.

    (c) Check that the *negatively* oriented parallelograms are precisely those with negative area.

27. Construct a non-zero vector perpendicular to each of the pairs of vectors A, C in question 21.

28. Calculate the (absolute value of the) area of the parallelogram determined by each of the pairs of vectors in question 21.

29. Determine the cosine of the angle between each of the following pairs of vectors:

    (a) $(1, 1, -2)$ and $(3, 0, 1)$;

    (b) $(1, 1, 1)$ and $(1, 1, -1)$;

    (c) $(1, 1, 1)$ and $(3, 3, 1)$.

30. For each of the pairs of vectors V and W given below, find k so that $V + kW \perp W$, or explain why no such k exists.

    (a) $V = (1, 1)$, $W = (2, -1)$.

    (b) $V = (1, 1, -2)$, $W = (3, 0, 1)$.

    (c) $V = (1, 1, -2)$, $W = (-4, -4, 8)$.

31. Find a non-zero vector orthogonal to each of the following pairs:

    (a) $V = (1, 1, -2)$, $W = (3, 0, 1)$.

    (b) $V = (1, 0, 0)$, $W = (0, 1, 0)$.

    (c) $V = (1, 1, 0)$, $W = (0, 1, 1)$.

32. Calculate the signed length of the projection of A on (the line in the direction of) C, where

    (a) $A = (1, 1, -4)$, $C = (3, 0, 1)$;

    (b) $A = (1, 1, 0)$, $C = (0, 1, 1)$;

    (c) $A = (2, 0, -2)$, $C = (2, 2, 0)$;

    (d) $A = (5, -1, -2)$, $C = (1, 1, 1)$.

33. For each pair of vectors A, C in the previous question, let $\mathcal{P}_C$ be the plane through the origin perpendicular to C. Determine whether A is on the same side of $\mathcal{P}_C$ as C, on the opposite side, or on $\mathcal{P}_C$ itself.

34. Calculate the absolute length of the projection of A on the line perpendicular to the vectors V and W, where

    (a) $A = (1, 1, 1)$; $V = (1, 1, -2)$, $W = (3, 0, 1)$.

    (b) $A = (1, 0, 1)$; $V = (1, 1, 0)$, $W = (0, 1, 1)$.

    (c) $A = (-1, 1, 2)$; $V = (1, 0, 1)$, $W = (2, 1, 1)$.

35. Sketch the oriented parallelepiped determined by the three vectors $(2, 0, 0)$, $(1, 2, 0)$, and $(1, 0, 3)$. Determine the signed volume of this parallelepiped. Is the orientation of this parallelepiped *positive* or *negative*?

36. Sketch the oriented parallelepiped determined by the three vectors $(2, 0, 0)$, $(1, -2, 0)$, and $(1, 0, 3)$. Determine the signed volume of this parallelepiped. Is the orientation of this parallelepiped *positive* or *negative*?

37. Determine the signed volume of the oriented parallelepiped determined by each of the following oriented triples of vectors.

    (a) $(1, 1, 1)$, $(1, 1, -2)$, $(3, 0, 1)$.

    (b) $(1, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$.

    (c) $(-1, 1, 2)$, $(1, 0, 1)$, $(2, 1, 1)$.

38. Let $A = (a_1, a_2, \ldots, a_n)$, $B = (b_1, b_2, \ldots, b_n)$, $C = (c_1, c_2, \ldots, c_n)$ be vectors in $\mathbb{R}^n$, and let k be an arbitrary real number. Show by direct calculation that the following facts are true.

    (a) $A \cdot (B + C) = A \cdot B + A \cdot C$.

    (b) $A \cdot B = B \cdot A$.

    (c) $(kA) \cdot B = k(A \cdot B)$.

39. Let A, B, and C be as in the previous question. The vector that displaces A to B displaces C to _____.

40. (a) Determine the lengths of $P = (1, 1, 2, 0)$ and $Q = (0, 2, -1, 0)$.

    (b) Determine the cosine of the angle between P and Q.

    (c) Determine the signed length of the projection of P on Q.

    (d) Determine the signed length of the projection of Q on P.

(e) Are these two signed lengths equal in value? Should they be?

(f) Determine k so that $P + kQ \perp Q$, or explain why no such k can be found.

41. In $\mathbb{R}^4$, the sets $X = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\}$ and $U = \{(0, 0, u, v) \mid u, v \in \mathbb{R}\}$ are two-dimensional planes. Show that $X \perp U$ in the sense that every vector in X is perpendicular to every vector in U.