Calculus III: Taylor polynomials worksheet

1. (a) Determine the normal to the surface S given by $x^2 - 3y^2 - z = 0$ at the point $(x, y, z) = (1, -2, -11)$. Then determine the equation of the tangent plane to S at this point.

   (b) Translate your result in previous part (a) to give the formula $z = ax + by + c$ for the tangent plane to the graph of $z = x^2 - 3y^2$ at $(x, y) = (1, -2)$.

   (c) Determine the first-order Taylor polynomial for the function $f(x, y) = x^2 - 3y^2$ at the point $(x, y) = (1, -2)$. Compare this result with your answer in part (b).

   (d) Generalize this in the following way: Write the equation of the tangent plane to the graph of $z = g(x, y)$ at an arbitrary point $(x, y) = (a, b)$.

2. Complete the square to show that the given function has a critical point of the indicated type at the origin. For example, $4x^2 - 12xy + 8y^2$ has a minimax at the origin, because it equals $(2x - 3y)^2 - y^2$, the sum of a positive square and a negative square.

   (a) $x^2 + 6xy + 10y^2$; minimum

   (b) $x^2 - 6xy + 8y^2$; minimax

   (c) $9x^2 - 18xy + y^2$; minimax

   (d) $-x^2 - 4xy - 6y^2$; maximum

   (e) $-x^2 - 4xy - 3y^2$; minimax

3. What kind of critical point does the given function have at the origin?

   (a) $100x^2 - xy + y^2$

   (b) $(x - 2y)(x + y)$

   (c) $-x^2 + 72xy + 9y^2$

4. Write the second-order Taylor polynomial for the given function at the given point.

   (a) $e^x \sin y$ at $(0, 0)$

   (b) $\cos x \cos y$ at $(0, \pi/2)$

   (c) $x^3 - 3x + y^2$ at $(-1, 0)$

   (d) $\ln(x^3 + y)$ at $(1, 0)$

   (e) xyz at $(1, -2, 4)$

   (f) $1 - \cos\theta + \frac{1}{2}v^2$ at $(\pi, 0)$

5. Write the fourth-order Taylor polynomial of $(x^2 + y^2)^2 - (x^2 + y^2)$ at the point $(x, y) = (1/2, 1/2)$.

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6. Let $f(x,y) = 3x^2 - x^3 - y^2$.

   (a) Verify that $(0,0)$ and $(2,0)$ are critical points of f.

   (b) Find the second-order Taylor polynomial for f at $(2,0)$; call it $P(x,y)$. Graph together $f(x,y)$ and $P(x,y)$ on a small neighborhood of $(2,0)$—specifically, use $1.9 \leq x \leq 2.1$, $-.1 \leq y \leq .1$.

   (c) Does P have a critical point at $(2,0)$? What kind? Do the graphs show that P and f have the same type of critical point at $(2,0)$? What kind of critical point does f have at $(2,0)$?

   (d) Find the second-order Taylor polynomial for f at $(0,0)$; call it $Q(x,y)$. Graph together $f(x,y)$ and $Q(x,y)$ on a small neighborhood of $(0,0)$—specifically, use $-.1 \leq x \leq .1$, $-.1 \leq y \leq .1$.

   (e) What kind of critical point does Q have at $(0,0)$. Do the graphs show that Q and f have the same type of critical point at $(0,0)$? What kind of critical point does f have at $(0,0)$?

7. (a) Find all critical points of $f(x,y) = x^3 + y^3 - 3x - 12y$.

   (b) At each critical point P of f, construct the second-order Taylor polynomial T_P of f. Does $T_P(x,y)$ *also* have a critical point at P? What kind?

   (c) In a small neighborhood of each of the critical points P, sketch the graph of f together with the Taylor polynomial T_P. Does f resemble T_P near P? Is P the same type of critical point for f that it is for T_P?

   (d) Conclusion: list the critical points of f, and indicate the type of each.

8. (a) Find all critical points of $f(x,y) = x^3 - 3xy^2 - x^2 + 3y^2$.

   (b) At each critical point P of f, construct the second-order Taylor polynomial T_P of f. Does $T_P(x,y)$ *also* have a critical point at P? What kind?

   (c) In a small neighborhood of each of the critical points P, sketch the graph of f together with the Taylor polynomial T_P. Does f resemble T_P near P? Is P the same type of critical point for f that it is for T_P?

   (d) Conclusion: list the critical points of f, and indicate the type of each.

9. Locate the critical point of $Q(x,y) = ax^2 + 2bxy + cy^2 + dx + ey + k$ and determine its type. (The type depends on the values of a, b and c, but *not* on d, e, or k.

10. Locate all the critical points of $\Phi(\theta, v) = 1 - \cos\theta + \frac{1}{2}v^2$, and determine the type of each.