Calculus III: planetary motion

In 1600, Johannes Kepler deduced from extensive astronomical data that the orbit of each planet is an ellipse with the sun at one focus. Before the end of the century, Isaac Newton had created a theory of motion and an incomparable tool—calculus—that could explain Kepler's law. What follows below is a summary of Newton's work in modern form.

Assume that the sun is at the origin of a coordinate system, and that the planet is at the point $\mathbf{x}(t)$ at time t. The goal is to show that the curve $\mathbf{x}(t)$ is an ellipse.

Newton's laws

Our starting points are Newton's laws of motion and law of universal gravitation:

Motion An object with no forces acting on it experiences no acceleration: $\mathbf{x}'' = 0$. Any force on an object induces an acceleration that is proportional to the force in both magnitude and direction. In other words, the force *vector* and the acceleration *vector* are proportional. The proportionality constant is called the *mass* m of the object:

$$\text{Force} = m\,\mathbf{x}''.$$

Gravitation Each object in the universe attracts every other object with a force whose magnitude is proportional to the product of the masses of the two objects and inversely proportional to the square of the distance between them. The force points in the direction of the attraction. The constant of proportionality G is the same for all objects.

Since the planet is at the position $\mathbf{x}$ and the gravitational force points back to the sun at the origin, the unit vector in this direction is $-\mathbf{x}/\|\mathbf{x}\|$. Therefore, if M is the mass of the sun, the sun's gravitational force on the planet can be written as

$$\text{Force} = -\frac{GMm}{\|\mathbf{x}\|^2}\frac{\mathbf{x}}{\|\mathbf{x}\|} = -\frac{GMm}{r^3}\,\mathbf{x}, \quad \text{where } r = \|\mathbf{x}\|.$$

Equating the forces given by the two different laws (and lumping G and M together as μ), we obtain, finally, the **differential equation of motion of a planet**:

$$\mathbf{x}'' = -\frac{\mu}{r^3}\,\mathbf{x}.$$

Notice that this equation does not depend on the mass m of the object, and it ignores the gravitational effects of the other planets.

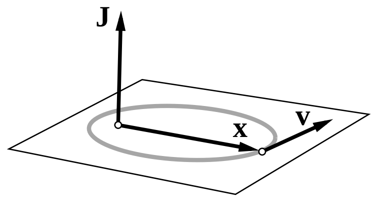
Angular momentum

The first step in solving the differential equation—which will show the orbit is an ellipse—is to use it to show that the orbit lies in a plane. Let $\mathbf{v} = \mathbf{x}'$ be the velocity vector and let

$\mathbf{J}(t) = \mathbf{x}(t) \times \mathbf{v}(t)$. Then, using the product rule and the differential equation, we find

$$\mathbf{J}' = \mathbf{x}' \times \mathbf{v} + \mathbf{x} \times \mathbf{v}' = \mathbf{v} \times \mathbf{v} - \frac{\mu}{r^3}\,\mathbf{x} \times \mathbf{x} \equiv 0.$$

In other words, the vector function $\mathbf{J}(t)$ is a *constant*. We call $\mathbf{J}$ the **angular momentum** (per unit mass) of the planet. In particular, $\mathbf{x}(t)$ lies in the plane perpendicular to $\mathbf{J}$, so all of the motion of the planet takes place in that plane.



Rotating the original coordinate system, if necessary, we can assume that the plane is the (x, y)-plane. Let (r, θ) be polar coordinates in this plane, so that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Since x and y are functions of t, so are r and θ. To determine the shape of the orbit it will be enough to determine how r depends on θ, that is, to determine the function $r = r(\theta)$. To start, we have

$$\begin{aligned}
\mathbf{x} &= (r\cos\theta, r\sin\theta, 0),\\
\mathbf{v} &= (r'\cos\theta - r\theta'\sin\theta, r'\sin\theta + r\theta'\cos\theta, 0),\\
\mathbf{J} &= \mathbf{x} \times \mathbf{v} = (0, 0, r^2\theta').
\end{aligned}$$

Because the vector $\mathbf{J}$ is a constant, its magnitude $J = r^2\theta'$ must be, as well.

The equation of motion in polar coordinates

Since $r^2 = \mathbf{x}\cdot\mathbf{x}$, and since r and $\mathbf{x}$ are functions of t, differentiation using the chain rule and the product rule shows that $2rr' = 2\mathbf{x}\cdot\mathbf{x}'$, or $rr' = \mathbf{x}\cdot\mathbf{v}$. Differentiation of $rr' = \mathbf{x}\cdot\mathbf{v}$ then shows (using $\mathbf{v}\cdot\mathbf{v} = v^2$)

$$rr'' + (r')^2 = \mathbf{v}\cdot\mathbf{v} + \mathbf{x}\cdot\mathbf{v}' = v^2 - \frac{\mu}{r^3}\,\mathbf{x}\cdot\mathbf{x} = v^2 - \frac{\mu}{r}.$$

The last step uses the original equation of motion; in fact, we can take what we have just discovered, namely $rr'' + (r')^2 = v^2 + \mu/r$, as a reformulation of the equation of motion. Next, we note that for any vectors $\mathbf{p}$ and $\mathbf{q}$, it is true that $(\mathbf{p}\cdot\mathbf{q})^2 + \|\mathbf{p}\times\mathbf{q}\|^2 = \|\mathbf{p}\|^2\|\mathbf{q}\|^2$. Therefore,

$$(\mathbf{x}\cdot\mathbf{v})^2 + \|\mathbf{x}\times\mathbf{v}\|^2 = (rr')^2 + J^2 = r^2v^2,$$

and thus $v^2 = (r')^2 + J^2/r^2$. Using this in the refomulated equation of motion, we find

$$rr'' + (r')^2 = (r')^2 + \frac{J^2}{r^2} - \frac{\mu}{r}, \qquad \text{or just} \qquad rr'' = \frac{J^2}{r^2} - \frac{\mu}{r}.$$

In the last equation r is treated as a function of t and r' means dr/dt. However, to describe the orbit we shall want to think of r as a function of θ: $r = r(\theta)$. This is not a problem because θ is itself a function of t, so we have $r = r(\theta(t))$ and, by the chain rule,

$$r' = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}.$$

With this observation, we can now convert the equation of motion

$$rr'' = \frac{J^2}{r^2} - \frac{\mu}{r}$$

into polar coordinates. In turns out, though, that the polar coordinate equation will be easier to understand if we work with the reciprocal radius, $u = 1/r$, instead of r itself. That is, $u = 1/r(\theta(t))$ and therefore

$$\begin{aligned} r &= \frac{1}{u(\theta(t))}, \\ r' &= \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -r^2\theta'\frac{du}{d\theta} = -J\frac{du}{d\theta}, \\ r'' &= \frac{dr'}{dt} = -J\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -J\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -J\frac{d^2u}{d\theta^2}\theta' = -J\frac{d^2u}{d\theta^2}\frac{J}{r^2} = -J^2u^2\frac{d^2u}{d\theta^2}. \end{aligned}$$

Therefore, in terms of u and θ, the equation of motion becomes

$$\frac{1}{u}\left(-J^2u^2\frac{d^2u}{d\theta^2}\right) = J^2u^2 - \mu u,$$

or just

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{J^2}.$$

The solution

If μ were zero, a solution to the last equation would simply be a function whose second derivative is the negative of itself—for example, the cosine function or any multiple of it. When $\mu \neq 0$, we need only add μ/J^2 to a solution for $\mu = 0$:

$$u(\theta) = A\cos\theta + \frac{\mu}{J^2}.$$

If we set $e = AJ^2/\mu$ (so $A = \mu e/J^2$) and then $k = J^2/\mu$, our solution takes the simple form

$$u = \frac{\mu}{J^2}(1 + e\cos\theta), \qquad \text{or} \qquad r = \frac{k}{1 + e\cos\theta}.$$

It remains to recognize this as an ellipse. Rewrite the last equation as

$$r + er\cos\theta = r + ex = k, \qquad \text{or} \qquad r = k - ex.$$

Now square this to get $r^2 = x^2 + y^2 = k^2 - 2kex + e^2x^2$, and rewrite the last as

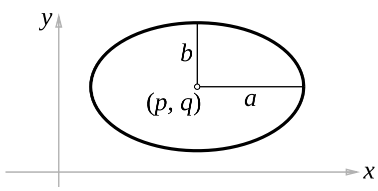
$$x^2 - e^2x^2 + 2kex + y^2 = (1-e^2)x^2 + 2kex + y^2 = (1-e^2)\left(x^2 + 2\frac{ke}{1-e^2}x\right) + y^2 = k^2.$$

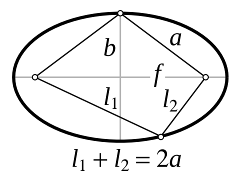
Now "complete the square in x" by adding $(1-e^2)\cdot\left(\frac{ke}{1-e^2}\right)^2$ to both sides:

$$(1-e^2)\left(x + \frac{ke}{1-e^2}\right)^2 + y^2 = k^2 + \frac{k^2e^2}{1-e^2} = \frac{k^2}{1-e^2}.$$

The standard form of an ellipse centered at $(x, y) = (p, q)$ with semimajor and semiminor axes a and b (assuming $a \geq b$) is

$$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1.$$





Geometrically, an ellipse is the locus of points whose distances l_1 and l_2 to two fixed points, its *foci*, have a fixed sum. By taking a point on the major axis, we see that $l_1 + l_2 = 2a$. By taking a point on the minor axis, we see $l_1 = l_2 = a$ and $a^2 = b^2 + f^2$, where f is the distance from the center to either focus. Thus $f = \sqrt{a^2 - b^2}$.

To put the solution we have above in standard form we need merely divide both sides of the equation by $k^2/(1-e^2)$ to make the right-hand side equal 1:

$$\frac{1}{k^2/(1-e^2)^2}\left(x + \frac{ke}{1-e^2}\right)^2 + \frac{1}{k^2/(1-e^2)}\,y^2 = 1.$$

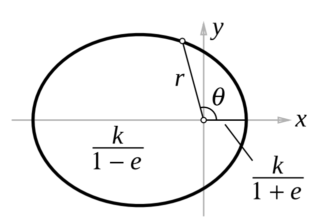
If $0 < e < 1$, this is the equation of an ellipse with

$$\text{center: } \left(\frac{-ke}{1-e^2}, 0\right), \quad \text{semimajor axis: } \frac{k}{1-e^2}, \quad \text{semiminor axis: } \frac{k}{\sqrt{1-e^2}}.$$

The distance from the center to a focus is

$$f = \sqrt{\frac{k^2}{(1-e^2)^2} - \frac{k^2}{1-e^2}} = \sqrt{\frac{k^2}{(1-e^2)^2} - \frac{k^2(1-e^2)}{(1-e^2)^2}} = \sqrt{\frac{k^2e^2}{(1-e^2)^2}} = \frac{ke}{1-e^2}.$$

Since the center is at $x = -ke/(1-e^2)$, one focus is at the origin. The ellipse is closest to the origin (the sun!) when $\theta = 0$ and $r = k/(1+e)$; it is farthest from the origin when $\theta = \pi$ and $r = k/(1-e)$. The nearest and farthest points are called the **perihelion** and **aphelion**.



Exercises

1. Suppose $\mathbf{x}(t) = (r\cos\theta, r\sin\theta, 0)$, where r and θ are (unknown) functions of t, and suppose also that $\mathbf{v} = \mathbf{x}'$, and $\mathbf{J} = \mathbf{x} \times \mathbf{v}$. Show

$$\begin{aligned} \mathbf{v} &= (r'\cos\theta - r\theta'\sin\theta, r'\sin\theta + r\theta'\cos\theta, 0), \\ \mathbf{J} &= \mathbf{x} \times \mathbf{v} = (0, 0, r^2\theta'). \end{aligned}$$

2. Show that $(\mathbf{p} \cdot \mathbf{q})^2 + \|\mathbf{p} \times \mathbf{q}\|^2 = \|\mathbf{p}\|^2\|\mathbf{q}\|^2$ for any vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^3$.

3. Carry out all the algebraic steps that convert the differential equation

$$rr'' = \frac{J^2}{r^2} - \frac{\mu}{r}$$

into

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{J^2},$$

using $u = 1/r(\theta(t))$ and

$$\begin{aligned} r &= \frac{1}{u(\theta(t))}, \\ r' &= \frac{dr}{dt} = -\frac{1}{u^2}\,\frac{du}{d\theta}\,\frac{d\theta}{dt} = -r^2\theta'\,\frac{du}{d\theta} = -J\,\frac{du}{d\theta}, \\ r'' &= \frac{dr'}{dt} = -J\,\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -J\,\frac{d^2u}{d\theta^2}\,\frac{d\theta}{dt} = -J\,\frac{d^2u}{d\theta^2}\,\theta' = -J\,\frac{d^2u}{d\theta^2}\,\frac{J}{r^2} = -J^2u^2\,\frac{d^2u}{d\theta^2}. \end{aligned}$$

4. We needed to assume $0 < e < 1$ in order to have the solution equation determine an ellipse. This question will explore what happens if $e > 1$. Then $k^2/(1-e^2)$ is a negative number, say $-b^2$, while $k^2/(1-e^2)^2$ remains positive, say a^2. If we let $ke/(1-e^2) = -p$, then the solution equation takes the form

$$\frac{(x-p)^2}{a^2} - \frac{y^2}{b^2} = 1.$$

   (a) This is the locus of what kind of curve? To answer, first consider the locus

$$x^2 - y^2 = 1, \qquad \text{and then} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

   [Answer: The orbit is a hyperbola. Since $e = AJ^2/\mu$, we will have $e > 1$ when the angular momentum J is sufficiently large. Thus, an object can *escape* from the sun (on a hyperbolic orbit) if it has a high enough angular momentum. We explore this further in question 6, below.]

(b) Show that the perihelion distance is still given by the formula $D = k/(1+e)$ (so the perihelion point is $(x, y) = (k/(1+e), 0)$).

5. What happens if $e = 1$ exactly? To answer this question, go back to the equation of the orbit in the form

$$x^2 + y^2 = k^2 - 2kex + e^2x^2 = k^2 - 2kx + x^2.$$

Show that this is a parabola whose nearest point to the origin (its *perihelion*) is at $(x, y) = (k/2, 0)$. Make a sketch of the orbit that shows this. (Note that $k/(1+e) = k/2$ in this case, so the perihelion distance is $k/(1+e)$ for *all* orbit shapes.)

6. The last two questions establish that an object escapes from the sun when $e \geq 1$ (on either a parabolic or hyperbolic orbit) but is forever trapped (on an elliptical orbit) if $e < 1$. This question shows that these possibilities depend only on the object's angular momentum $J \neq 0$ and its perihelion distance $D = k/(1+e)$.

   Assume that the object is on an escape orbit, so $e \geq 1$. Using the fact that $k = J^2/\mu$, show that

$$D \leq \frac{J^2}{2\mu},$$

   and hence that $J \geq \sqrt{2\mu D}$. This is the condition that guarantees an object will escape from the sun, never to return.

7. What is the shape of the orbit if $e = 0$? What is the relation between angular momentum, J, and perihelion distance, D, in this case?

8. What is the shape of the orbit if the angular momentum, J, is zero? Show that θ is constant, implying the object moves on a radial line either directly towards or away from the sun.

9. There is another function, like angular momentum, that is a "constant of the motion". Show that the **energy**

$$E(t) = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mathbf{v} \cdot \mathbf{v}}{2} - \frac{\mu}{\|\mathbf{x}\|}$$

   is constant when $\mathbf{x}(t)$ is a solution to the equation of motion. (In fact, E is more properly called the energy *per unit mass*—just as with J.)