Calculus III: Lines and planes worksheet

1. (a) Sketch in (x, y)-plane the line $(x, y) = (2 - 2t, 1 + 3t)$. Use $-6 \leq x \leq 6$, $-6 \leq y \leq 6$.

   (b) What is the slope of this line?

2. What is the slope of $s(4, -5) + (1, 2)$ as a line in $\mathbb{R}^2$?

3. Express in parametric form $tD + A$ the line

   (a) in the direction $(1, 1)$ passing through the point $(1, -1)$;

   (b) in the direction $(5, 1, 2)$ passing through the point $(-1, 4, 2)$;

   (c) passing through the points $(2, -1, 3)$ and $(2, 3, 1)$;

   (d) parallel to the line $(5 - s, 5 + 2s)$ passing through the origin;

   (e) parallel to the line $(5 - u, 5 + 2u, 7u, 1 - 2u)$ passing through $(-1, 2, -2, 0)$;

   (f) orthogonal to the line $(5 - r, 5 + 2r)$ passing through $(-1, 2)$.

4. Write in parametric form $tD + A$ the line that passes through $(2, 1)$ when $t = 0$ and through $(4, 4)$ when $t = 2$.

5. Write in parametric form $tD + A$ the line that passes through $(2, 1)$ when $t = 3$ and through $(4, 4)$ when $t = -1$.

6. Determine which of the following pairs of lines are parallel; in each case, give your reason.

   (a) $t(-1, 1, 4) + (3, 7, 2)$ and $t(1, 6, 4) + (3, 7, 2)$.

   (b) $t(-2, 1, 1) + (4, 1, 8)$ and $t(-2, 1, 1) + (6, 1, 9)$.

   (c) $t(7, 2) + (2, 3)$ and $(9, -1) - t(21, 6)$.

   (d) $(5 + 3t, 4 - t)$ and $(5 + t, 4 - 3t)$.

   (e) $(3s + 1, 4 - s)$ and $(1 - 15s, 5s)$

7. How are the lines $(3 - 6t, 4 + 4t, 1 + 8t)$ and $(9u, 4 - 6u, 5 - 12u)$ related?

8. How are the lines $(3 - 6t, 4 + 4t, 1 + 8t)$ and $(9u, 6 - 6u, 5 - 12u)$ related? The answer to this question is different from the preceding: these lines are even more closely related. What is the difference?

9. Write the equation of the plane in (x, y, z)-space that

   (a) is perpendicular to $(1, 1, 1)$ and passes through $(1, 0, 0)$;

   (b) is orthogonal to $(4, 1, -3)$ and passes through $(1, 4, -3)$;

(c) has normal $(1/\sqrt{2}, -1/\sqrt{2}, 0)$ and passes through $(\pi, -1, e)$.

(d) is parallel to the plane $x - 2y + z = 7$ and passes through the origin.

(e) is orthogonal to the line $(5 - 2t, 5 + t, 6)$ and passes through $(5, 5, 6)$.

10. Write, in the form $ax + by = c$, the equation of the line in $\mathbb{R}^2$ that

    (a) is orthogonal to $(1, 2)$ and passes through $(5, 7)$;

    (b) is parallel to $(1, 2)$ and passes through $(5, 7)$;

    (c) is given parametrically as $(t + 1, t - 1)$;

    (d) is orthogonal to $4x + 3y = 5$ and passes through the origin;

    (e) contains the points $(-2, 3)$ and $(5, -1)$.

11. Write (using coordinates (x, y, u, v) in $\mathbb{R}^4$) the equation of the hyperplane

    (a) through the origin and normal to $(1, 2, 3, 4)$;

    (b) through $(1, 2, 3, 4)$ and normal to $(1, -1, 2, -2)$;

    (c) through $(1, 2, 3, 4)$ and parallel to the hyperplane $x - y + 3u - v = 1$.

12. Write a set of parametric equations

$$(x, y, u, v) = (at + b, ct + d, et + f, gt + h)$$

    for the line normal to the hyperplane $x - y + 3u - v = 1$ and passing through the point $(0, 0, 1, 0)$.

13. Using coordinates $X = (x_1, x_2, \ldots, x_k)$ in $\mathbb{R}^k$, write the equation of the hyperplane with normal $N = (n_1, n_2, \ldots, n_k)$ passing through the point $A = (a_1, a_2, \ldots, a_k)$.

14. Let $X = (x_1, x_2, x_3, x_4, x_5)$, and let $A = (2, 1, -3, 7, 0)$, $B = (0, 2, 1, 0, 3)$. Describe, in geometric language, the set of points X in $\mathbb{R}^5$ that satisfy the equation $A \cdot (X - B) = 0$.

15. Determine a unit normal vector for

    (a) $4x + 7y - 2z = 19$;

    (b) $4x + 7y = -2$;

    (c) $(x - 2) + (y - 3) + (4 - z) = 0$;

    (d) $5u - v + 4w + 2y - z = 32$ in (u, v, w, x, y, z)-space $\mathbb{R}^6$;

    (e) $a_1(x_1 - b_1) + a_2(x_2 - b_2) + \cdots + a_k(x_k - b_k) = 0$ in $\mathbb{R}^k$.

16. What is the (cosine of the) angle between the line $(x, y, z) = (1 + t, 1 - t, 2t)$ and the normal to the plane $x + y + 3z = 5$?

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17. What is the (cosine of the) angle between the line $(x, y) = (2 - t, 3 + 3t)$ and the line $2x + y = 5$? (Note: this concerns the angle between the lines themselves, not between one line and the normal to the other.)

18. What is the (cosine of the) angle between the line $(x, y, z, w) = (t, 1 + t, 2 - t, 3)$ and the hyperplane $x + y + 3z - w = 0$? (Note: again, this concerns the plane itself, *not* the normal to the plane.)

19. Write the equation of the plane that contains the origin and the pair of vectors

    (a) $(1, 0, 0)$ and $(0, 1, 0)$;

    (b) $(1, 1, -2)$ and $(3, 0, 1)$.

20. Write the equation of the plane that is parallel to the plane containing the origin and the vectors $(2, -1, 3)$ and $(1, 1, 1)$, and that passes through the point $(0, 4, 2)$.

21. Suppose A and B are given vectors, and P is a given point, in $\mathbb{R}^3$. Write, in terms of the variable point X, the equation of the plane that is parallel to the plane containing the origin and the vectors A and B, and that passes through the point P.

22. Consider the plane $\mathcal{P} : x - y + 2z = 5$ in $\mathbb{R}^3$.

    (a) Show that $A = (x, y, z) = (1, -2, 1)$ lies in $\mathcal{P}$.

    (b) Determine a normal N to $\mathcal{P}$; then determine the length of the projection of A on (the line in the direction of) N.

    (c) Suppose X is an arbitrary point in $\mathcal{P}$; show that the length of the projection of every X on N is the same as the length of the projection of A on N.

23. Determine the (perpendicular) distance from the origin to

    (a) the plane $x + y + z = 4$ in $\mathbb{R}^3$;

    (b) the plane $2x - y + 3z = -20$ in $\mathbb{R}^3$;

    (c) the line $5x + 2y = 7$ in $\mathbb{R}^2$;

    (d) the hyperplane $2x_1 + 3x_2 - x_3 + 5x_4 = 1$ in $\mathbb{R}^4$.

24. Use ordinary Euclidean geometry to find the distance from the origin to the line $5x + 2y = 7$.

25. Determine the distance between the parallel planes

$$2x + y - 4z = 9 \qquad \text{and} \qquad 2x + y - 4z = 8.$$

26. Determine the distance between the planes

$$6(x-2)+9(y-1)-30(z+5)=0 \qquad \text{and} \qquad z=\frac{4x+6y+7}{20}.$$

27. Write the equations of the two planes that are 3 units on either side of the plane $x+y+z=0$.