Calculus III: gradients worksheet

1. The **gradient** of $f(x, y)$ is the vector of partial derivatives grad $f = \nabla f = (f_x(x, y), f_y(x, y))$. Determine the gradient grad $f = \nabla f$ of

   (a) $f(x, y) = x^2 + y^3$;

   (b) $f(x, y) = \sin(xy)$;

   (c) $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$;

   (d) $f(u, v) = u^4 - u^2v^2 + v^4$.

   (e) $f(p, q, r) = \ln \dfrac{pq}{r}$.

2. Determine the direction D in which the function $z = f(X)$ is increasing most rapidly at the point A, when

   (a) $f(x, y) = x^2 - y^2$, $A = (2, -1)$;

   (b) $f(x, y) = x^3 + y^3 - x - y$, $A = (0, 1)$;

   (c) $f(x, y) = \cos(xy)$, $A = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$;

   (d) $f(x, y) = x^2 + y^2$, $A = (a, b)$ arbitrary;

   (e) $f(x, y, z) = xyz$, $A = (1, -1, 0)$;

   (f) $f(x, y, u, v) = xy + uv$, $A = (0, 1, 0, -2)$.

3. Use $D_{\mathbf{u}}f(P) = \text{grad}\, f(P) \cdot \mathbf{u}$ to recalculate the directional derivative $D_{\mathbf{u}}f(P)$ in each of the following cases, and compare your answer with what you did in question 12 on the Derivatives worksheet. Normalize (i.e., convert into a *unit* vector) the direction vector $\mathbf{u}$, as necessary.

   (a) $f = xy$, $P = (2, 1)$, $\mathbf{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$;

   (b) $f = e^x \cos y$, $P = (0, \pi)$, $\mathbf{u} = (-1, 0)$;

   (c) $f = e^x \cos y$, $P = (0, \pi)$, $\mathbf{u} = (1, 0)$;

   (d) $f = x^3 - xy^2 - 2x + 1$, $P = (1, 0)$, $\mathbf{u} = (1, 0)$;

   (e) $f = x^3 - xy^2 - 2x + 1$, $P = (1, 0)$, $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$;

   (f) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $\mathbf{u} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$;

   (g) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$;

   (h) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $\mathbf{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$;

   (i) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $\mathbf{u} = (3, 4)$;

   (j) $f = x^3 - 3xy^2$, $P = (0, 0)$, $\mathbf{u} = (\alpha, \beta)$ arbitrary;

   (k) $f = xy \sin z$, $P = (1, 3, \pi/2)$, $\mathbf{u} = (2, 1, -1)$.

4. Determine all points (x, y) where the gradient of the following function $f(x, y)$ is zero.

   (a) $f(x, y) = x^2 + y^3 - 3y$;

   (b) $f(x, y) = x^3 + y^3 - 3(x + y)$;

   (c) $f(x, y) = \sin x \sin y$;

   (d) $f(x, y) = x^3 - 3xy^2 - x^2 + 3y^2 = (x - 1)(x^2 - 3y^2)$;

5. Make a sketch of the **gradient vector field** $\operatorname{grad} f(x, y) = \nabla f(x, y)$ of the function $f(x, y)$ given below over the domain

$$-2 \le x \le 2 \qquad -2 \le y \le 2.$$

   In particular, note all points where the gradient is the zero vector, and make the all arrows only one-quarter or one-fifth of their true lengths (so the arrows don't clutter up your diagram). Draw enough arrows to make the overall field clear.

   (a) $f(x, y) = xy$.

   (b) $f(x, y) = 3y - y^3 - x^2$.

6. Determine $\nabla f = \operatorname{grad} f$ for

   (a) the general linear form in n variables,

$$f(X) = A \cdot X = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n;$$

   (b) the quadratic form in two variables,

$$f(x, y) = ax^2 + 2bxy + cy^2,$$

   or

$$f(x_1, x_2) = a_{11} x_1^2 + a_{12} x_1 x_2 + a_{21} x_2 x_1 + a_{22} x_2^2, \qquad (a_{21} = a_{12});$$

   (c) the quadratic form in n variables,

$$f(X) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j, \qquad a_{ji} = a_{ij}.$$

   (Suggestion: first see what this looks like when $n = 2$ and $n = 3$.)

7. For each of the following functions f,

   (i) determine the *direction* of most rapid increase of the function at the given point;

   (ii) determine the *rate* of most rapid increase at that point;

(iii) using (i) and (ii), estimate the value of the function one unit away from the given point in the direction of most rapid increase.

(a) $f(x, y) = xy$ at $(1, 2)$.

(b) $f(x, y) = x^2 + y^2$ at $(3, 4)$.

(c) $f(u, v) = \cos uv$ at $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$.

(d) $f(z, w) = z^3 + w^3 - z - w$ at $(0, 1)$.

(e) $f(x, y, z) = xyz$ at $(1, 2, 3)$.

8. Make a representative sketch of the gradient vector field and, on the same coordinate plane, a sketch of level curves of the following functions $z = f(x, y)$.

(a) $z = x$

(b) $z = 3y$

(c) $z = 2x - 3y$

(d) $z = 2x - 3y + 7$

(e) $z = x^2$

(f) $z = x^2 + 4y^2$

(g) $z = x^2 - 4y^2$

(h) $z = xy$

(i) $z = x^2 y$

(j) $z = x^2 y^2$

(k) $z = x^2 + y^4$

(l) $z = \sin xy$

(m) $z = x - y^2$

(n) $z = 3x^2 + 6xy + y^2$

9. Using the evidence of the gradient field and the family of level curves you have just found, sketch the graph of each of the functions in the preceding question.

10. Find all critical points (i.e., points where the gradient vector is zero) of each of the following functions.

(a) $f(x, y) = 2x^2 + 7xy + y^2$

(b) $f(x, y) = 2x^2 + 7xy - y^2$

(c) $f(x, y) = x^3 + y^3 - x - y$

(d) $f(p, q) = 2pq$

(e) $f(u, v) = u^3 - 3uv^2 - u^2 + 3v^2 = (u - 1)(u^2 - 3v^2)$

(f) $f(x, y) = \sin(\pi xy)$

(g) $f(r, \theta) = r \cos \theta$

(h) $f(x, y) = ax^2 + 2bxy + cy^2 + px + qy + r$

11. Give an example of a function $f(x, y, z)$ whose gradient is the constant $\nabla f = (2, -1, 3)$.

12. Give an example of a function $f(u, v)$ whose gradient is $\nabla f = (u^2, -v)$.

13. Give two different examples of functions $f(x, y)$ whose gradient is $\nabla f = (y, x)$.

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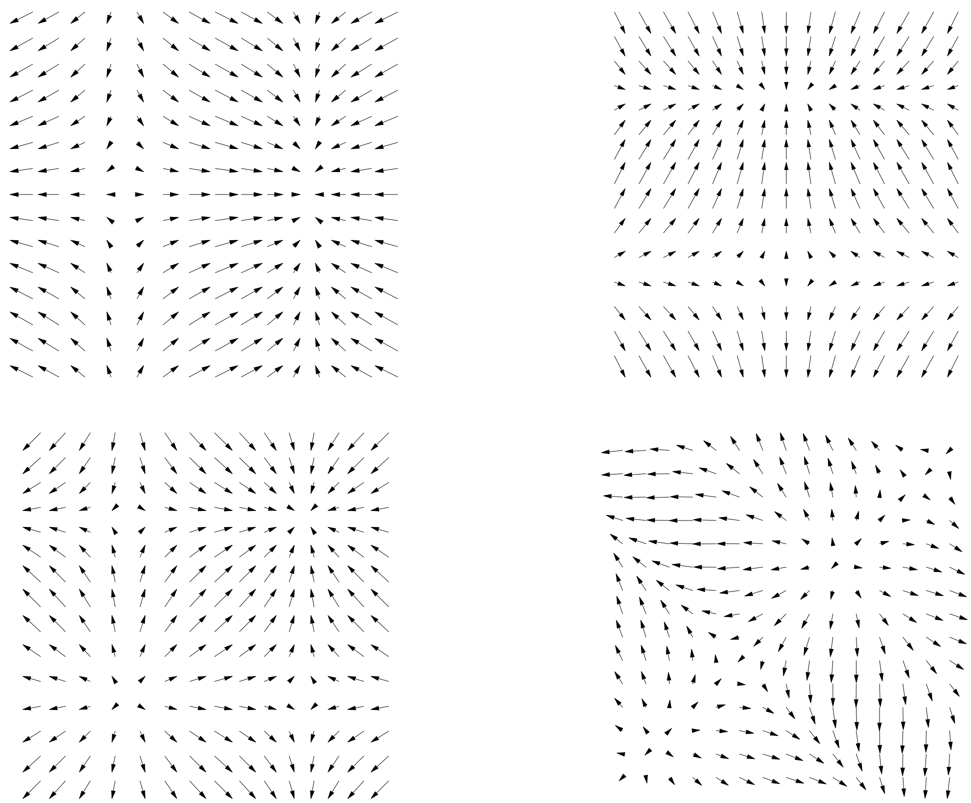
14. Sketch the level curves of the function whose gradient field is shown below. Mark the location of the highest values of the function with an "H" and the lowest with an "L".



15. For each curve given below by an equation of the form $G(x, y) = c$, find a normal at the given point and then write the equation of the tangent line (in the form $Ax+By = C$) to the curve at that point.

    (a) $x^2 + y^2 = 25$ at $(3, -4)$
    (b) $x^2 - y^2 = 9$ at $(5, 4)$
    (c) $y^2 - x^2(x + 1) = 0$ at $(1, \sqrt{2})$

16. (a) Show that the circle $x^2 + y^2 = 25$ and the ellipse $16x^2 + 9y^2 = 288$ intersect at the point $(3, -4)$.
    (b) Determine the angle of intersection of these two curves at $(3, -4)$. (Suggestion: consider their normals.)

17. For each surface given below by an equation of the form $G(x, y, z) = c$, find a normal at the given point and then write the equation of the tangent plane to the surface at that point.

    (a) $x^2 + y^2 + z^2 = 1$ at $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$
    (b) $x^2 + y^2 - z^2 = 1$ at $(1, 1, 1)$
    (c) $xyz = 27$ at $(9, -3, -1)$