Calculus III: derivatives worksheet

1. Determine $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when

   (a) $f(x,y) = x^2 - xy - 2y^2$;

   (b) $f(x,y) = e^x \cos y$;

   (c) $f(x,y) = \arctan\left(\dfrac{y}{x}\right)$;

   (d) $f(x,y) = \dfrac{\sin x}{1+y^2}$.

2. Let $f(x,y) = x^2 - xy - 2y^2$ and let

$$G(x) = f(x,1) \qquad \text{and} \qquad H(y) = f(2,y).$$

   (a) Sketch the graphs $z = G(x)$ and $z = H(y)$.

   (b) Determine the derivatives $G'(x)$ and $H'(y)$.

   (c) Compare $G'(2)$ with $\dfrac{\partial f}{\partial x}(2,1)$ and $H'(1)$ with $\dfrac{\partial f}{\partial y}(2,1)$.

3. Let $\varphi(u,v) = u^2 v - v^3$ and let

$$G(u) = \varphi(u,b) \qquad \text{and} \qquad H(v) = \varphi(a,v),$$

   where a and b are some constants.

   (a) Determine the derivatives $G'(u)$ and $H'(v)$.

   (b) Compare $G'(a)$ with $\dfrac{\partial \varphi}{\partial u}(a,b)$ and $H'(b)$ with $\dfrac{\partial \varphi}{\partial v}(a,b)$.

4. Determine f_x, f_y, f_z, f_{xy}, f_{yy}, and f_{yz} for $f(x,y,z) = e^x y^2 \sin z$.

5. Determine the partial derivatives f_1, f_2, f_{11}, f_{12}, and f_{22} for $f(p,q) = p^2 q^3$.

6. Find *all* partial derivatives of $f(x,y) = x^3 - 3xy^2 + x^2 + y^2$.

7. Suppose $f(x,y) = g(x) \cdot h(y)$; find f_{xy} in terms of g and h and their derivatives.

8. Suppose $f(x,y) = F(x-y)$; show $f_x + f_y \equiv 0$.

9. The **Laplacian** of $f(x,y,z)$ is defined to be

$$\Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

   The particular sum of second derivatives represented by Δ is call the **Laplace operator**. We shall meet other such differential operators in the following.

   (a) Determine the Laplacian Δf when $f = xy + yz + zx$.

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(b) Determine the Laplacian Δf when $f = (xy)^2 + (yz)^2 + (zx)^2$.

(c) Show that $\Delta g \equiv 0$ when g is a linear function: $g(x, y, z) = ax + by + cz + d$.

10. Obtain the formula for the function $G(t) = f(P + tD)$ when

(a) $f = xy$, $P = (2, 1)$, $D = (1, \sqrt{3})$;
(b) $f = e^x \cos y$, $P = (0, \pi)$, $D = (1, 0)$;
(c) $f = x^3 - xy^2 - 2x + 1$, $P = (1, 0)$, $D = (1, 0)$;
(d) $f = x^3 - xy^2 - 2x + 1$, $P = (1, 0)$, $D = (1, 1)$;
(e) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $D = (-1, 1)$;
(f) $f = x^3 - 3xy^2$, $P = (0, 0)$, $D = (\alpha, \beta)$ arbitrary;
(g) $f = xy \sin z$, $P = (1, 3, \pi/2))$, $D = (2, 1, -1)$.

11. For each of the functions $G(t)$ you obtained in the preceding question (for part (f), let $\alpha = \beta = 1$), make a sketch of the graph of $y = G(t)$ for t near 0 and indicate whether the slope $G'(0)$ is *positive*, *negative*, or *zero*.

12. Obtain the directional derivative $D_{\mathbf{u}} f(P)$, when

(a) $f = xy$, $P = (2, 1)$, $\mathbf{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$;
(b) $f = e^x \cos y$, $P = (0, \pi)$, $\mathbf{u} = (-1, 0)$;
(c) $f = e^x \cos y$, $P = (0, \pi)$, $\mathbf{u} = (1, 0)$;
(d) $f = x^3 - xy^2 - 2x + 1$, $P = (1, 0)$, $\mathbf{u} = (1, 0)$;
(e) $f = x^3 - xy^2 - 2x + 1$, $P = (1, 0)$, $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$;
(f) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $\mathbf{u} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$;
(g) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$;
(h) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $\mathbf{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$;
(i) $f = x^3 + y^3 - x - y$, $P = (0, 0)$, $\mathbf{u} = (3, 4)$;
(j) $f = x^3 - 3xy^2$, $P = (0, 0)$, $\mathbf{u} = (\alpha, \beta)$ arbitrary;
(k) $f = xy \sin z$, $P = (1, 3, \pi/2))$, $\mathbf{u} = (2, 1, -1)$.

13. What is the rate of increase of $z = x^2 - 3y^2$ at the the point $(5, 1)$ in the direction $\mathbf{u} = (1/\sqrt{5}, -2/\sqrt{5})$?

14. What is the rate of increase of $w = xu + yv$ at the point $(x, y, u, v) = (1, -1, 0, 2)$ in the direction $\mathbf{u} = (1/3, 0, 2/3, -2/3)$?

15. What is the rate of increase of $t = 4r - 3s$ at the point (a, b) in the direction of the unit vector $\mathbf{u} = (\alpha, \beta)$ (i.e., $\alpha^2 + \beta^2 = 1$)?