Calculus III: curves worksheet

1. Using graph paper and a calculator to determine values, make accurate plots of each of the following curves in $\mathbb{R}^2$. Note that what you draw is the *track* of the curve in each case. Describe the curve, or track, when it has a simple form that you can give a name to. (For example, the first curve is an ellipse.)

   (a) $(3\cos t, \sin t)$, $0 \le t \le 2\pi$.

   (b) $(\cos 4t, \sin 4t)$, $0 \le t \le 2\pi$.

   (c) (t^2, t^3), $-2 \le t \le 2$.

   (d) $\left(\dfrac{2t}{1+t^2}, \dfrac{1-t^2}{1+t^2}\right)$, $-3 \le t \le 3$.

   (e) $(t\cos t, t\sin t)$, $0 \le t \le 2\pi$.

   (f) $(t/\pi + \cos t, \sin t)$, $0 \le t \le 6\pi$.

2. (a) Obtain the velocity and acceleration vectors for each of the curves in the preceding question.

   (b) For which of these curves is the velocity vector at every given point perpendicular to the acceleration at that point?

3. (a) On the plots you have already drawn for 1(a) and 1(b), make accurate sketches of the velocity and acceleration vectors at (the points where) $t = 0$ and $t = \pi/4$.

   (b) On the plots you have already drawn for 1(c) and 1(d), make accurate sketches of the velocity and acceleration vectors at $t = -1$, $t = 0$, and $t = 1$.

   (c) On the plots you have already drawn for 1(e), make accurate sketches of the velocity and acceleration vectors at $t = 0$, $t = \pi/2$, and $t = \pi$.

   (d) In each case, is the velocity vector that you drew *tangent* to its curve at the point where you placed it? (It should be!)

   (e) Do the acceleration vectors have a similar consistent relation to their curves—e.g., are they all *tangent* to their curves, or are they all *normal* to them?

4. How does the curve in 1(b) differ from the curve $(\cos t, \sin t)$? (To answer this question, compare the tracks of the two curves and compare how the parameter point moves in the two cases.) How do their velocities differ?

5. The curve $(a + r\cos t, b + r\sin t)$ is a familiar one; give a complete description of it in words.

6. (a) Make a sketch of the curve $x(t) = \sin t$, $y(t) = 2\sin t$, for $0 \le t \le 4\pi$.

   (b) What is the "track" for this curve? Notice that $y = 2x$; how is this fact manifested in your sketch?

(c) As t varies from 0 to 4π, the parameter point $(x(t), y(t))$ moves back and forth over the track of this curve how many times?

(d) When (i.e., for which values of t) is the velocity vector equal to zero? At what points on the curve does this happen? Note that the velocity becomes zero when the parameter point is "turning around" on the track. Is this what you would expect?

7. Let D and A be arbitrary vectors in $\mathbb{R}^2$. Determine the velocity and acceleration vectors for the curve $X(t) = tD + A$.

8. Consider the curve $Y(t) = \left(\dfrac{2t}{1+t^2}, \dfrac{1-t^2}{1+t^2} \right)$ of question 1(d), but now allow t to take all possible real values: $-\infty < t < \infty$.

(a) Determine $\|Y(t)\|^2$. This should confirm the observation you made about the shape of the track of $Y(t)$ in your answer to question 1. Why?

(b) Show the following four limits exist, and determine their values:

$$\lim_{t\to-\infty} \frac{2t}{1+t^2}, \qquad \lim_{t\to-\infty} \frac{1-t^2}{1+t^2},$$

$$\lim_{t\to\infty} \frac{2t}{1+t^2}, \qquad \lim_{t\to\infty} \frac{1-t^2}{1+t^2}.$$

With these limits, we can define $Y(-\infty)$ to be

$$Y(-\infty) = \lim_{t\to-\infty} Y(t) = \left(\lim_{t\to-\infty} \frac{2t}{1+t^2}, \lim_{t\to\infty} \frac{1-t^2}{1+t^2} \right),$$

and similarly for $Y(\infty) = \lim_{t\to\infty} Y(t)$. In fact, your work should show that $Y(-\infty) = Y(\infty)$. in other words, the two limit points (exist and) are equal.

(c) Complete the following: The track of $Y(t)$ is a _____ minus the point _____, which the parameter point $Y(t)$ approaches as $t \to \pm\infty$.

(d) Make a sketch of $Y(t)$ that reflects all the information you now have; be sure your sketch indicates the location of $Y(\pm\infty)$.

9. (a) For each of the curves $X(t)$ in question 1, sketch the graph of the *speed* of the parameter point along the curve. (The **speed** of $X(t)$ is the length $\|X'(t)\|$ of the velocity vector; it is a scalar function of t.)

(b) Which curves in question 1 have *constant speed*?

10. Suppose the curve $X(t)$ has constant speed; that is, suppose $\|X'(t)\| = c$. Show that $X'(t) \perp X''(t)$ for each t. Suggestion: take $X'(t) \cdot X'(t) = c^2$ (why?) and differentiate.

You have just proved a theorem that can be worded in the following way: *If a point moves along a curve with constant speed, its acceleration is always* normal *to the curve.* Compare this result with what you observed in your answer to question 2. Notice, furthermore, that your result does not depend on the *dimension* of the space containing the curve; the theorem holds for curves in any $\mathbb{R}^n$.

11. (a) Sketch, in $\mathbb{R}^3$, the curve $X(t) = (\cos t, \sin t, t)$. Describe this curve in words, as well, so that you don't have to rely entirely on your ability to draw in three dimensions.
    (b) Calculate X' and X'', and calculate the speed $\|X'\|$.
    (c) Add to your sketch in part (a) the velocity and acceleration vectors $X'(0)$ and $X''(0)$ at the point $X(0)$. Is $X'(0)$ tangent to the curve at that point; is $X''(0)$ perpendicular?
12. Sketch the space curve $x = t$, $y = t^2$, $z = t^3$, for $-1 \le t \le 1$.
13. Write parametric equations $x = \varphi(t)$, $y = \psi(t)$ that describe a circle of radius 3 centered at the point $(1, -2)$.
14. Write parametric equations for the ellipse that is centered at the origin and passes through the points $(\pm 5, 0)$ and $(0, \pm 3)$.
15. (a) Write parametric equations $X(t) = tD + A$ for a straight line that passes through the points $(1, 0, -1)$ and $(2, 2, 0)$.
    (b) What is the speed of the parameter point in your parametrization in part (a)?
    (c) Construct a new parametrization whose speed is three times that of the parametrization in part (a).
16. Write parametric equations of the circle of radius 3 centered at the origin so that the parameter point moves with constant speed 15. Write another set of parametric equations of the same circle so that the speed is 1.
17. Determine the **arc length** of each of the following curves over the specified interval.
    (a) $(4t - 2, 5 - t)$, $-1 \le t \le 1$.
    (b) $(\cos 2t, \sin 2t)$, $0 \le t \le \pi$.
    (c) $(\cos 2t, \sin 2t)$, $0 \le t \le \pi/2$.
    (d) $(\cos 4t, \sin 4t)$, $0 \le t \le \pi/2$.
    (e) (t^3, t^2), $1 \le t \le 4$.
    (f) $(\sin^2 t, \cos^2 t)$, $0 \le t \le \pi/2$. Sketch this curve, too.
    (g) $(e^t \cos t, e^t \sin t)$, $0 \le t \le \pi/2$.
    (h) $(\cos t, \sin t, kt)$, $0 \le t \le 2\pi$; k is a constant.

(i) $\left(\dfrac{2t}{1+t^2}, \dfrac{1-t^2}{1+t^2}\right)$, $-\infty < t < \infty$.
It may look intractable, but you can, in fact, calculate the integral here. Even before you do, though, your familiarity with the curve should tell you that the arc length is 2π.

18. For each of the following curves, obtain (i) the **arc length parameter** s and (ii) the **unit speed** (or **arc length**) **parametrization** $Z(s)$. In each case, verify that $\|Z'(s)\| \equiv 1$.

    (a) $(3t+2, 1-4t)$, starting from $t=7$.

    (b) $(\sin^2 t, \cos^2 t)$, from $t=0$.

    (c) $(\cos 3t, \sin 3t)$, from $t=0$.

    (d) $(3\cos t, 3\sin t)$, from $t=0$.

    (e) $(r\cos t, r\sin t)$, from $t=0$; r is a positive constant.

    (f) (t^3, t^2), from $t=0$.

    (g) $(4+4\cos 3t, -1+4\sin 3t)$, from $t=\pi/6$.

    (h) $(\cos^3 t, \sin^3 t)$, from $t=0$.

    (i) $(e^t \cos t, e^t \sin t)$, from $t=0$.

    (j) $(\cos t, \sin t, kt)$, from $t=0$; k is a positive constant.

19. For each of the curves in the preceding question (for which you therefore already have the unit speed parametrization $Z(s)$, determine

    (a) the **curvature vector** $K(s) = Z''(s)$;

    (b) the **curvature function** $\kappa(s) = \|K(s)\|$;

    (c) the **radius of curvature** $\rho(s) = \dfrac{1}{\kappa(s)}$.

    Note: "κ" and "ρ" are the Greek letters *kappa* and *rho*; they correspond to the English "k" and "r".