1. (a) Let $x = u^3 - 3uv^2$, $y = 3u^2v - v^3$. Determine $\dfrac{\partial(x,y)}{\partial(u,v)}$.

   (b) Express $x^2 + y^2$ as a function of u and v.

   (c) Determine $\dfrac{\partial(u,v)}{\partial(x,y)}$ as a function of x and y.

2. (a) Make a careful sketch of the region D in the first quadrant that is bounded by the x-axis and the parabolas

$$x = \frac{y^2}{4} - 1, \qquad x = 1 - \frac{y^2}{4}, \qquad x = 4 - \frac{y^2}{16}.$$

   (b) Show that D is the image of a rectangle $a \le u \le b$, $0 \le v \le c$ in the (u,v)-plane under the "quadratic" map

$$\mathbf{q} : \begin{cases} x = u^2 - v^2, \\ y = 2uv. \end{cases}$$

   In particular, determine the values of a, b, and c.

   (c) Use $\mathbf{q}$ to make a change of coordinates that will allow you to determine

$$\iint_D x\,dx\,dy.$$

   (d) Determine the area of D. (Can you use $\mathbf{q}$ here, too?)

   (e) Determine the average value μ of $f(x,y) = x$ on D.

   (f) Determine $\displaystyle\iint_D (x - \mu)\,dx\,dy$.

3. Compute $d\alpha$ and $d^2\alpha = d(d\alpha)$ for the 1-form α in (x,y,u,v)-space given by

$$\alpha = (x + y + u + v)\,dx + (xy - uv)\,du.$$

4. Let $\mathbb{V} = (x^2y, y^2z, z^2x)$.

   (a) Determine both the divergence and the curl of $\mathbb{V}$.

   (b) Determine $\operatorname{curl}(\operatorname{curl}\mathbb{V})$.

   (c) Determine $\operatorname{grad}(\operatorname{div}\mathbb{V})$.

5. (a) Let $\mathcal{T}$ be the triangle in $\mathbb{R}^3$ with vertices $(2, 2, 0)$, $(3, 3, 1)$, $(1, 4, 0)$; show that $N = (2, 1, -3)$ is normal to $\mathcal{T}$. Correction: 2 May
   (b) Determine the flux of the field $\mathbb{V} = (-2, 5, -1)$ through $\mathcal{T}$ in the direction of the normal N.

6. Let $\vec{S}$ be the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$, oriented by its outward normal. Determine
$$\iint_{\vec{S}} \frac{1}{x}\,dy\,dz + \frac{1}{y}\,dz\,dx + \frac{1}{z}\,dx\,dy.$$

7. Determine the flux of the field $\mathbb{V} = (x, 2y, 3z)$ *out* of the sphere of radius R centered at the origin.

8. Determine the circulation of the field $\mathbb{W} = (z - y, z, x + y)$ around the path $(x, y, z) = (\cos t, \sin t, \sin 2t)$, $0 \le t \le 2\pi$.

9. A flow $\mathbb{V}$ is said to be **irrotational** if $\operatorname{curl}\mathbb{V} = 0$, and is said to be **incompressible** if $\operatorname{div}\mathbb{V} = 0$.
   (a) Give an example of a *non-zero, non-linear* irrotational flow.
   (b) Give an example of a *non-zero, non-linear* incompressible flow.
   (c) Can a non-constant flow be both irrotational and incompressible? Explain.

10. (a) The image T of the map $\mathbf{t} : \vec{U} \to \mathbb{R}^3 : (\theta, \varphi) \to (x, y, z)$,
$$\mathbf{t} : \begin{cases} x = (a + r\cos\varphi)\cos\theta, \\ y = (a + r\cos\varphi)\sin\theta, \\ z = r\sin\varphi, \end{cases} \qquad \vec{U} : \begin{array}{l} 0 \le \theta \le 2\pi, \\ 0 \le \varphi \le 2\pi, \end{array}$$
   is a **torus** that is a curved tube of radius r around a core circle of radius a. Make a careful sketch of T when $r = 2$ and $a = 5$.
   (b) The parametrization $\mathbf{t}$ gives T an orientation; determine the orienting normal and the area of $\vec{T}$ as functions of a and r.

11. Suppose the closed oriented surface $\vec{S}$ in $\mathbb{R}^3$ is the complete boundary of the solid region $\vec{R}$ (i.e., $\partial\vec{R} = \vec{S}$), and the orienting unit normal of $\vec{S}$ is $\mathbf{n}$. Suppose also that the vector $\mathbf{x}$ points from the origin to an arbitrary point on $\vec{S}$. Show that
$$\frac{1}{3}\iint_{\vec{S}} \mathbf{x}\cdot\mathbf{n}\;dA = \text{volume}(R).$$